Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 23, 2017, No. 1, 1–3

A note on the quartic Diophantine equation $A^4 + hB^4 = C^4 + hD^4$

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Received: 19 August 2016 **Accepted:** 31 January 2017

Abstract: Integer solutions of the diophantine equation $A^4 + hB^4 = C^4 + hD^4$ are known for all positive integer values of $h < 1000$. While a solution of the aforementioned diophantine equation for any arbitrary positive integer value of h is not known, Gerardin and Piezas found solutions of this equation when h is given by polynomials of degrees 5 and 2, respectively. In this paper, we present several new solutions of this equation when h is given by polynomials of degrees $2,\ 3$ and 4.
Keywords: Biquadrates, Fourth powers.
AMS Classification: 11D25.

This paper is concerned with the quartic diophantine equation,

$$A^4 + hB^4 = C^4 + hD^4, \tag{1}$$

where h is a given integer. Numerical solutions of Eq. (1) for various values of h were recorded by several authors ([2, pp. 647–648]. A table of solutions for 75 integer values of $h \leq 101$ was compiled by Choudhry [1]. This table was first extended by Piezas [3] to include all positive integer values of $h \leq 101$, and then by Tomita [5] for all positive integer values of $h < 1000$ except $h = 967$. The missing solution for $h = 967$ was supplied by Bremner (as mentioned by Tomita) and now solutions of (1) are known for all positive integers $h < 1000$. These numerical solutions confirm Choudhry's remark [1] that Eq. (1) seems to have a solution in integers for all positive integer values of h.

There is no known method that would yield a solution in integers of (1) for any arbitrary value of h. When $h = 2p^3(p^2 - 1)$, Gerardin had noted the solution $A = 2p^2,\ B = p - 1,\ C =$

$2p,\ D = p + 1$ (as quoted by Dickson [2, p. 647]). Piezas has made a remark [4] from which it immediately follows that a solution of Eq. (1) when $h = p^2 - 3$ is given by

$$A = p^3 + 2p^2 - 3p - 2, \quad B = p^3 - p - 2,$$

$$C = p^3 - 2p^2 - 3p + 2, \quad D = p^3 - p + 2.$$

Further, Tomita [5] has noted that when $h = p^4 + q^4$, an obvious solution of Eq. (1) is given by $A = p^2,\ B = q,\ C = q^2,\ D = p$.

In Table 1 we present several new solutions of (1) in which h is given by simple polynomials of degrees $2,\ 3$ and 4. These solutions were obtained by experimenting with Eq. (1) and may be readily verified by hand or by using any symbolic computation software such as MAPLE or Mathematica. We also note that when one nontrivial solution of (1) has been found, infinitely many integer solutions may be obtained following a procedure described in [1].

Table 1: **Solutions of the equation** $A^4 + hB^4 = C^4 + hD^4$

h	A	B	C	D
$p^2 + 2$	$p^3 + 2p + 1$	$p^2 - p + 1$	$p^3 + 2p - 1$	$p^2 + p + 1$
$p(p^2 + 4)$	$p - 2$	2	$p + 2$	0
$p^4 - 1$	p	0	1	1
$2p^4 - 2$	$p^2 + 2p - 1$	$p - 1$	$p^2 - 2p - 1$	$p + 1$
$p^4 + 3p^2 + 1$	$p^2 + p + 1$	$p - 1$	$p^2 - p + 1$	$p + 1$

We note that if a solution of Eq. (1) when $h = \phi(p)$ is given by $A = A(p),\ B = B(p),\ C = C(p),\ D = D(p)$, then a rational solution of (1) when $h = q^4\phi(p/q)$ is given by $A = qA(p/q),\ B = B(p/q),\ C = qC(p/q),\ D = D(p/q)$, and integer solutions may be obtained by multiplying through by a constant. Thus, from Table 1, we readily obtain solutions of (1) when the value of h is given by any of the polynomials $(p^2 + 2q^2)q^2,\ pq(p^2 + 4q^2),\ 8pq(p^2 + q^2),\ p^4 - q^4,\ 2(p^4 - q^4)$ and $p^4 + 3p^2q^2 + q^4$.

Acknowledgment

I wish to thank the Harish-Chandra Research Institute, Allahabad for providing me with all necessary facilities that have helped me to pursue my research work in mathematics.

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