Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 22, 2016, No. 4, 78–81

Infinite product involves the Tribonacci numbers

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Received: 16 March 2016 **Accepted:** 30 October 2016

Abstract: In this short note, we discuss the integer part for the inverse of

$$1-\prod_{k=n}^{\infty}\left(1-\frac{1}{T_k}\right),$$

where T_n are the Tribonacci numbers. We also consider a similar formula for the Tribonacci numbers with indices in arithmetic progression and give an open problem of the Diophantine equation about the Tribonacci numbers.

Keywords: Tribonacci number, Infinite product, Diophantine equation.
AMS Classification: 11B39, 11D99.

1 Introduction

The sequences of Fibonacci numbers $\{F_n\}_{n=0}^{\infty}$ ([3, A000045]) and Tribonacci numbers $\{T_n\}_{n=0}^{\infty}$ ([3, A000073]) are defined, respectively, by

$$F_{n+1}=F_n+F_{n-1},\ F_0=0,\ F_1=1$$
$$T_{n+2}=T_{n+1}+T_n+T_{n-1},\ \ T_0=0,\ T_1=T_2=1.$$

Ohtsuka [2] gave the solution of his problem as

$$\left\lfloor\left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{F_k}\right)\right)^{-1}\right\rfloor=F_{n-2}\quad(n\geq 3),$$

where $\lfloor\cdot\rfloor$ is the floor function.

Naturally, a question arises: is there a similar formula for the Tribonacci numbers? In this note, we shall provide such formula.

2 Results

We first provide two lemmas which will be used in the proof of main result.

Lemma 1. *Let n be a positive integer. Then*

(i) $T_n^2 - T_{n-1}T_{n+1} = T_{-(n+1)}$

(ii) $T_n > T_{-(n+3)} \quad (n \geq 3)$.

See the proof of Lemma 1 in [1].

Lemma 2. *Let $n \geq 3$ be a positive integer. Then*

$$\frac{T_n - T_{n-1} - 1}{T_{n+1} - T_n - 1} \cdot \frac{T_{n+1} - T_n}{T_n - T_{n-1}} \leq \frac{T_n - 1}{T_n} < \frac{T_n - T_{n-1}}{T_{n+1} - T_n} \cdot \frac{T_{n+1} - T_n + 1}{T_n - T_{n-1} + 1}. \tag{1}$$

Proof. The left inequality (1) is equivalent to

$$T_n(T_n - T_{n-1} - 1)(T_{n+1} - T_n) \leq (T_n - 1)(T_n - T_{n-1})(T_{n+1} - T_n - 1),$$

or

$$T_{n+1}T_{n-1} - T_n^2 + T_n - T_{n-1} \geq 0.$$

Using the identities in Lemma 1, it is clear that the above inequality is correct for $n \geq 3$. In a similar way, we can rewrite the right inequality (1) as

$$T_n^2 - T_{n+1}T_{n-1} + T_{n+1} - T_n > 0,$$

which is true. Hence, the inequality (1) is proved. □

Now we are ready to state and prove the main result.

Theorem 1. *Let $n \geq 3$ be a positive integer. Then*

$$\left\lfloor \left(1 - \prod_{k=n}^{\infty} \left(1 - \frac{1}{T_k} \right) \right)^{-1} \right\rfloor = T_n - T_{n-1}.$$

Proof. By the left inequality (1) for $n \geq 3$, we have

$$\frac{T_n - T_{n-1} - 1}{T_n - T_{n-1}} \leq \frac{T_n - 1}{T_n} \cdot \frac{T_{n+1} - T_n - 1}{T_{n+1} - T_n}.$$

By applying the above inequality repeatedly, we obtain

$$\begin{aligned}\frac{T_n - T_{n-1} - 1}{T_n - T_{n-1}} &\le \frac{T_n - 1}{T_n}\cdot\frac{T_{n+1}-1}{T_{n+1}}\cdot\frac{T_{n+2}-T_{n+1}-1}{T_{n+2}-T_{n+1}} \\ &\le \frac{T_n - 1}{T_n}\cdot\frac{T_{n+1}-1}{T_{n+1}}\cdot\frac{T_{n+2}-1}{T_{n+2}}\cdot\frac{T_{n+3}-T_{n+2}-1}{T_{n+3}-T_{n+2}} \\ &\vdots \\ &\le \prod_{k=n}^{\infty}\left(1-\frac{1}{T_k}\right).\end{aligned}$$

Similarly, using the right inequality (1), we have

$$\frac{T_n - T_{n-1}}{T_n - T_{n-1} + 1} > \frac{T_n - 1}{T_n}\cdot\frac{T_{n+1} - T_n}{T_{n+1} - T_n + 1}.$$

Repeating the above inequality, we obtain

$$\frac{T_n - T_{n-1}}{T_n - T_{n-1} + 1} > \prod_{k=n}^{\infty}\left(1-\frac{1}{T_k}\right).$$

Therefore,

$$\frac{T_n - T_{n-1} - 1}{T_n - T_{n-1}} \le \prod_{k=n}^{\infty}\left(1-\frac{1}{T_k}\right) < \frac{T_n - T_{n-1}}{T_n - T_{n-1} + 1},$$

or

$$T_n - T_{n-1} \le \left(1 - \prod_{k=n}^{\infty}\left(1-\frac{1}{T_k}\right)\right)^{-1} < T_n - T_{n-1} + 1.$$

Hence,

$$\left\lfloor \left(1 - \prod_{k=n}^{\infty}\left(1-\frac{1}{T_k}\right)\right)^{-1} \right\rfloor = T_n - T_{n-1} \quad (n \ge 3).$$

□

We can generalize the identity (i) of Lemma 1. It is easy to see that

$$\begin{aligned}T_{n-2}T_{n+2} - T_n^2 &= T_{-(n+2)} - T_{-n} \\ T_{n-3}T_{n+3} - T_n^2 &= -T_{-(n+5)} \\ T_{n-4}T_{n+4} - T_n^2 &= -4T_{-(n+5)}.\end{aligned}$$

The following analogous results are similarly obtained as that of Theorem 1.

Corollary 1. *Let $n \ge 2$ be a positive integer. Then*

(i) $$\left\lfloor \left(1 - \prod_{k=n}^{\infty}\left(1-\frac{1}{T_{2k}}\right)\right)^{-1} \right\rfloor = T_{2n} - T_{2n-2}$$

(ii) $$\left\lfloor \left(1 - \prod_{k=n}^{\infty}\left(1-\frac{1}{T_{2k-1}}\right)\right)^{-1} \right\rfloor = T_{2n-1} - T_{2n-3}$$

$$(iii)\quad \left\lfloor \left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{T_{3k}}\right)\right)^{-1}\right\rfloor = T_{3n}-T_{3n-3}$$

$$(iv)\quad \left\lfloor \left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{T_{4k}}\right)\right)^{-1}\right\rfloor = T_{4n}-T_{4n-4}.$$

An open problem for the Tribonacci numbers is to find the solutions of the Diophantine equation

$$T_m = T_{-n},$$

where m, n are positive integers greater than 1.

We claim that the all solutions (m, n) of the above equation are $(2, 2)$, $(3, 5)$, $(4, 8)$, $(5, 11)$, $(9, 30)$, $(15, 33)$ and $(17, 34)$.

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