Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 22, 2016, No. 4, 73–77

Landau's Fourth problem

J. V. Leyendekkers[1] and A. G. Shannon[2, 3]

[1] Faculty of Science, The University of Sydney, NSW 2006, Australia

[2] Emeritus Professor, University of Technology Sydney, NSW 2007, Australia

[3] Campion College, PO Box 3052, Toongabbie East, NSW 2146, Australia

e-mails: t.shannon@campion.edu.au, tshannon38@gmail.com

Received: 15 October 2015 **Accepted:** 30 October 2016

Abstract: Primes of the form $n^2 + 1$ show no deviations contrary to the natural integer structure within the modular ring Z_4 and the sum of two squares. Hence primes of this form should occur to infinity with other primes. Trend characteristics of primes and composites were compared graphically.

Keywords: Modular rings, Primes, Composites.
AMS Classification: 11A07, 11A51, 11B37.

1 Introduction

At the 1912 International Congress of Mathematicians, in Cambridge UK, Edmund Landau of Georg-August-Universität Göttingen in a paper entitled "Geloste und ungelöste Problème aus der Theorie der Primzahlverteilung und der Riemann'schen Zetafunktion" [1] listed four basic problems about primes. They are now known as Landau's problems. They are as follows (in slightly different terminology from that of Landau):

- Goldbach conjecture: Can every even integer greater than 2 be written as the sum of two primes?
- Twin prime conjecture: Are there infinitely many primes p such that $p + 2$ is prime?
- Legendre's conjecture: Does there always exist at least one prime between consecutive perfect squares?
- Landau question: Are there infinitely many primes of the form $n^2 + 1$?

As of 2016, all four problems are unresolved because of our limited knowledge of infinity, let alone gaps between consecutive primes and the need to attack them with asymptotic proofs [2]. Many experimental methods have been tried, including the comprehensive sieve approach of Shanks [3].

We consider the Landau question with a question: as there is an infinity of primes why should those of this form be limited? We explore this with integer structure [4] and show that the classes in the modular ring persist structurally intact to infinity so that any deviations from the structure by any particular types of primes would be obvious

2 The structure of $n^2 + 1$ in the modular ring Z_4

Integers of the form $(n^2 + 1) \in \bar{1}_4 \subset Z_4$ (Table 1).

Row $r_i \downarrow$	**Class** $i \rightarrow$	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$	**Comments**
0		0	1	2	3	$N = 4r_i + i$
1		4	5	6	7	even $\bar{0}_4$, $\bar{2}_4$
2		8	9	10	11	$(N^n, N^{2n}) \in \bar{0}_4$
3		12	13	14	15	odd $\bar{1}_4, \bar{3}_4$; $N^{2n} \in \bar{1}_4$

Table 1: Classes and rows for Z_4

Since n must be even and only Class $\bar{0}_4$ contains powers for even integers, then

$$N = n^2 + 1 = 4r_0 + 1 = 4r_1 + 1, \tag{2.1}$$

so that $N \in \bar{1}_4$. For example, $p = 5 = 4 \times 1 + 1$ and $2^2 + 1 = 4 \times 1 + 1$.

Now all primes in Class $\bar{1}_4$ are a sum of squares ($x^2 + y^2$), and unlike the composites, the primes have only one (x,y) pair that has no common factors. Thus

$$p = x^2 + y^2. \tag{2.2}$$

The values of x and y may be obtained from

$$x, y = \tfrac{1}{2}\left(A \pm \sqrt{2p - A^2}\right). \tag{2.3}$$

in which x is odd and y is even and $A = x + y, \sim \sqrt{2p}$ for primes. Thus

$$p = 2x^2 - 2Ax + A^2. \tag{2.4}$$

If $x = 1$, then

$$p = 1 + y^2, \tag{2.5}$$

Thus since $a = 4r_1 + 1$ which can always be a solution. This structure is invariant so that a prime of this form will be part of the structure to infinity. The invariance can be seen in the straight trend lines of Figure 1 with the data from Table 4.

3 Right-end-digit structure

This right-end-digit (RED) (an integer modulo 10) structure is consistent with that above. N^*, the RED of *N,* follows the pattern 5, 7, 7, 5, 1 (Table 2).

n*	**(n^2)***	**(n^2+1)***
`	0	1
2	4	5
4	6	7
6	6	7
8	4	5

Table 2: Even REDs

The REDs of $n^2, (n^2)^*$ can yield $(n^2+1)^* = 5,$ but such an *N* can never be a prime, other than 5, so *n** = 2 and 8 are excluded. For the interval *n* = 4 (*N* = 17) to *n* = 156 (*N* = 24337) there are 21 composites and 28 primes (excluding *N** = 5), but the primes have only one (*x*, *y*) pair whereas the composites have the same number of pairs as there are factors. In the interval considered here the composites have only two factors and hence two (*x*, *y*) pairs. Some examples are shown in Table 3, with some of the data from there displayed in Table 4 and Figure 1.

N	Type	(*x*,*y*)	*N*	Type	(*x*,*y*)	*N*	Type	(*x*,*y*)	*N*	Type	(*x*,*y*)
5	p	1,2	2117	29,73	1,46	6401	37,173	25,76	13457	p	1,116
17	p	1,4			31,34	7057	p	1,84	14401	p	1,120
37	p	1,6	2501	41,61	1,50	7397	13,569	1,86	15377	p	1,124
101	p	1,10			49,10			79,34	15877	p	1,126
197	p	1,14	2917	p	1,54	8101	p	1,90	16901	p	1,130
257	p	1,16	3137	p	1,56	8837	p	1,94	17957	p	1,134
401	p	1,20	3601	13,277	1,60	9217	13,709	1,96	18497	53,349	1,136
577	p	1,24			55,24			89,36			71,116
677	p	1,26	4097	17,29	1,64	10001	73,137	1,100	19601	17,1153	1,140
901	17,53	1,30			31,56			65,76			63,124
		15,26	4357	p	1,66	10817	29,373	1,104	20737	89,233	1,144
1157	13,89	1,34	4901	13,29	1,70			71,76			129,64
		31,14			49,50	11237	17,661	1,106	21317	p	1,146
1297	p	1,36	5477	p	1,74			49,94	22501	p	1,150
1601	p	1,40	5777	53,109	1,76	12101	p	1,110	23717	37,641	1,154
1937	13,149	1,44			41,64	12997	41,317	1,114			49,146
		41,16	6401	37,173	1,80			111,26	24337	p	1,156

Table 3: Composite pairs [Types are p: prime, or composite factors]

N	**A1 (1 + *y*)**	**A2 (*x* + *y*)**
901	31	41
1157	35	45
1937	45	57
2117	47	65
2501	51	59
3601	61	79
4097	65	87
4901	71	99
5777	77	105
6401	81	101
7397	87	113
9217	97	125
10001	101	141
10817	105	147
11237	107	143
12997	115	137
18497	137	187
19601	141	187
20737	145	193
23717	155	192

Table 4: *N* and (1 + *y*) [A1] and (*x* + *y*) [A2]

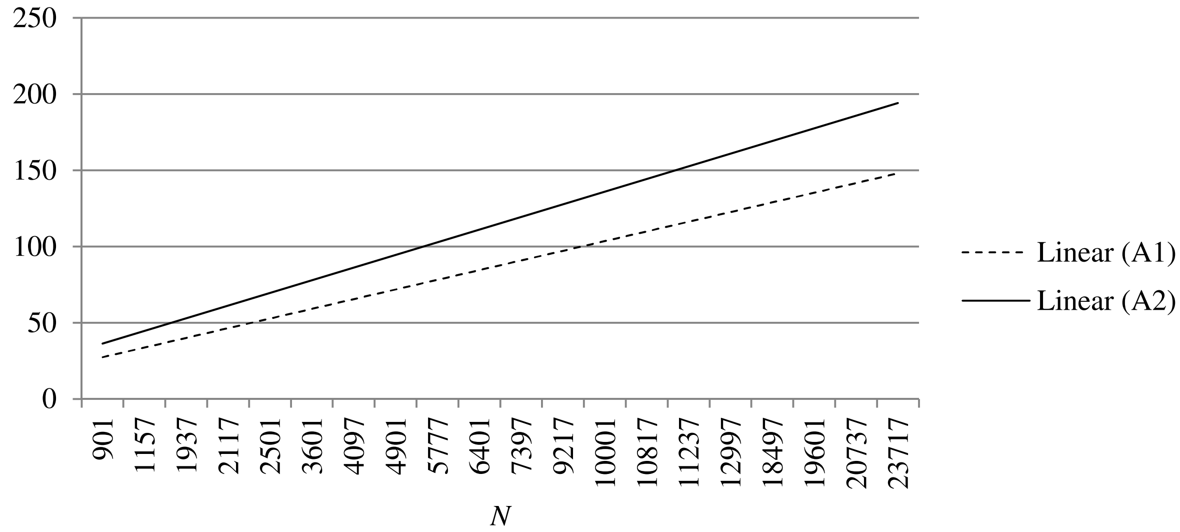

Figure 1: Linear trend lines of *N* vs (1 + *y*) [A1] and (*x* + *y*) [A2]

4 Final comments

The consistent structure indicates that primes and composites maintain their structures to infinity. Any unique characteristic is shown by the composites that have factors in class $\bar{1}_4$ and so will always have ordered pairs (x, y) accordingly. Naturally generalizations of the problem, such as $h.2^n + k$ have been explored in order to shed light on the problem [6, 7].

References

[1] Hardy, G. H., & Heilbronn, H. (1938) Edmund Landau. *Journal of the London Mathematical Society*, 13(4), 302–310.

[2] Cioabă, S. M., & Ram Murty, M. (2008) Expander graphs and gaps between primes. *Forum Mathematicum*, 20(4), 745–756.

[3] Shanks, D. (1959) A sieve method for factoring numbers of the form $n^2 + 1$. *Mathematical Tables and other Aids to Computation* (now *Mathematics of Computation*), 13, 78–86.

[4] Leyendekkers, J. V., Shannon, A.G., Rybak, J.M. (2007) *Pattern Recognition: Modular Rings and Integer Structure*. North Sydney: Raffles KvB Monograph No.9.

[5] Sloane, N.J.A. (1973+) *The On-Line Encyclopedia of Integer Sequences*. A002496.

[6] Inkeri, K., & Sirkesalo, J. (1959) Factorization of certain numbers of the form $h.2^n + k$. *Annales Universitasis Turkuensis, Series Ai*., 388.

[7] Riesel, H. (1994) *Prime Numbers and Computer Methods for Factorization*. Boston: Birkhäuser.