Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 22, 2016, No. 4, 62–72

Upper and lower bounds for π based on Vieta's geometrical approach

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Received: 9 October 2015 **Accepted:** 30 October 2016

Abstract: In this paper, we propose an upper and a lower bound of the number π expressed as the limit to infinity of two sequences. These sequences are constructed using geometric methods based on the Vieta's approach. As far as geometrical methods for computing π is concerned, numerical results are provided to show that the proposed result is comparable to the existing ones.

Keywords: Pi, Vieta's method, Telescopic series.
AMS Classification: 11B30.

1 Introduction

The number π is probably the most fascinating number in the history of mathematics. The interest in this number can be traced, as far back as the days of Archimedes, or even earlier, up to the present day. Many formulae and methods to calculate π have been derived; ranging from geometrical methods of Archimedes and Vieta in the early 250 BC to calculus based methods of Newton, Gauss and Euler in the late 1660s, to modular function based theory by Ramanujan in the early 1910s and to infinite expansion methods using products and radicals by Osler and Sato fairly recently in the early 2000 (see eg. [1, 2] and the list of references herein). All these late methods aims to provide fast algorithms that is capable to approximate π up to a very high accuracy; up to the millionth digit. A very thorough and detailed study of the history various formulae and algorithms for computing π is given in [1].

In this work, we focus on the early geometrical methods for finding π. The aim here is not to break the record of the latest digit of π but rather to complete some missing formula in the geometrical methods classification. For this, recall that the Archimedes method involves approximating π by the perimeters of polygons inscribed and circumscribed about a given circle. On the other hand, the Vieta's method of computing π consist in comparing the areas of regular polygons with 2^k and 2^{k+1} sides inscribed in a circle and which led to obtaining the following formula:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots .$$

The first term in the product, $\frac{\sqrt{2}}{2}$, is the ratio of areas of a square and an octagon, the second term is the ratio of areas of an octagon and a hexadecagon, etc. Based on the same spirit of Vieta, instead of comparing the areas of 2^k-gons inscribed and circumscribed about the unit circle (circle of radius 1) we iteratively compute the difference of areas of two successive 2^k-gons. So doing, we obtain two telescopic series that leads to providing an upper and a lower bound of π in terms of the limits to infinity of two sequences. As far as the author is aware, such geometrical results are not available in the literature.

In the next section, the main results of the work is presented in the form of two theorems followed by a corollary. Finally, some numerical results are presented and some conclusions are drawn.

Notation: Throughout this work, the Euclidean distance between two points P and Q will be denoted as $|PQ|$.

2 Main results

The first result of this section is summarized in the following theorem:

Theorem 1. *Let $(b_n)_{n\geq 0}$ be the sequence defined by:*

$$b_n = \sqrt{2}\sqrt{1-\sqrt{1-\left(\frac{b_{n-1}}{2}\right)^2}} \tag{1}$$

with $b_0 = 2$. Then,

$$\pi = \lim_{n\to+\infty} 2^n b_n.$$

Proof. Consider a circle of radius 1 in which a square, with side b_1, is inscribed inside it as depicted in Figure 1.

It is clear that the area of the circle is equal to π and that $b_1 = \sqrt{2}$. We denote by A_0 the area of this shaded square; that is $A_0 = \sqrt{2}\times\sqrt{2} = 2$. In fact, A_0 is a very rough approximation of the area of the circle.

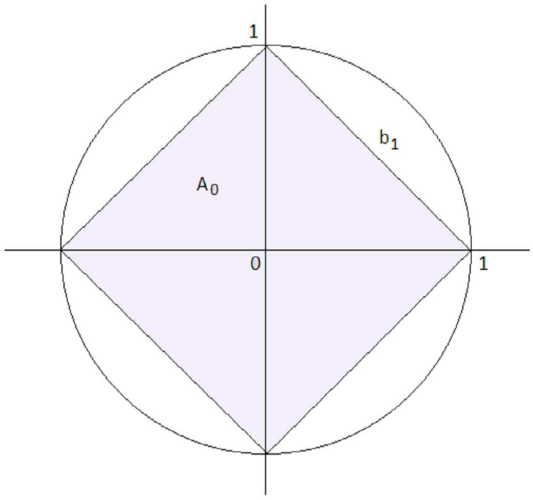

Figure 1: 1st approximation: A_0

Next, to obtain a better approximation of the area of the circle, we add, to A_0, the areas of the four isosceles triangles that are inscribed inside the four segments outside the square as depicted in the shaded part of Figure 2.

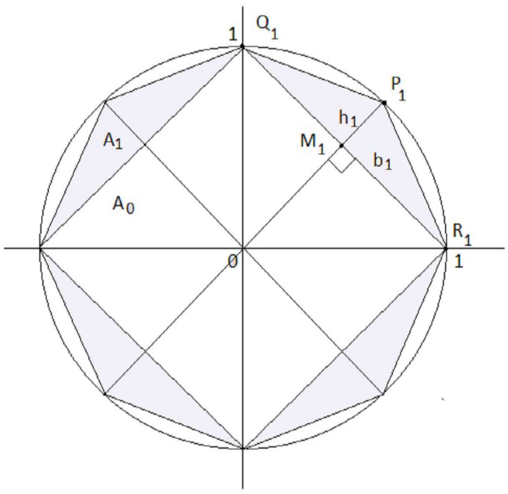

Figure 2: 2nd approximation: $A_0 + A_1$

These isosceles triangles have as base the sides of the square. Also, the line joining the centre of the circle and the vertex P_1 of the triangle cuts the side of the square perpendicularly at its midpoint M_1. Consequently, one can use the Pythagoras theorem to compute the areas of the shaded triangles. Indeed, one can easily check that the height of the triangle $Q_1P_1R_1$ is given by:

$$\begin{aligned} h_1 &= 1 - \sqrt{1 - \left(\frac{b_1}{2}\right)^2} \\ &= 1 - \sqrt{1 - \left(\frac{\sqrt{2}}{2}\right)^2} = 1 - \frac{\sqrt{2}}{2} \end{aligned}$$

Consequently, by denoting the areas of all the 4 triangles by A_1, we get

$$\begin{aligned} A_1 &= 4 \times \frac{1}{2} b_1 h_1 \\ &= 2 b_1 h_1 = 2 \times \left(\sqrt{2}\right) \times \left(1 - \frac{\sqrt{2}}{2}\right) = 0.82843. \end{aligned}$$

As a result, a better approximation of the area of the circle is:

$$\pi \approx A_0 + A_1 = 2.82843.$$

Similarly, to obtain a better approximation of the area of the circle, we inscribe a new set of isosceles triangles between the octagon and the circle, as shown in the shaded part of Figure 3. The base of these triangles are the sides of the octagon. Also, the vertices of these triangles are located at midpoint of each arc lying on the side of the octagon. As such, we obtain a 16-gons figure.

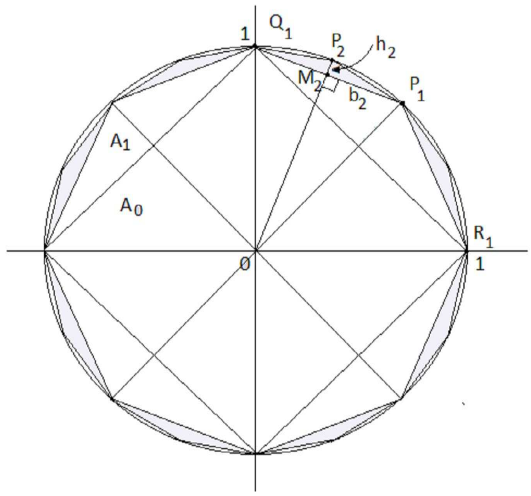

Figure 3: 3rd approximation: $A_0 + A_1 + A_2$

By using the Pythagoras theorem, one easily see that

$$b_2 = \sqrt{\left(\frac{b_1}{2}\right)^2 + (h_1)^2}.$$

Again, by applying the Pythagoras theorem and knowing that $|OP_2| = 1$, we have

$$h_2 = |M_2 P_2| = 1 - \sqrt{1 - \left(\frac{b_2}{2}\right)^2}.$$

Finally, the area of the 8 triangles is given by

$$A_2 = 8 \frac{b_2 h_2}{2} = 2^2 b_2 h_2 \simeq 0.23304.$$

Consequently, the 3rd approximation of π is given as: $\pi \simeq A_0 + A_1 + A_2 = 2.82843 + 0.23304 = 3.06147$.

By applying the same procedure as above, one can determine A_n for all $n \in \mathbb{N}$ as follows:

$$A_n = 2^n b_n h_n \tag{2}$$

$$b_n = \sqrt{\left(\frac{b_{n-1}}{2}\right)^2 + (h_{n-1})^2} \tag{3}$$

$$h_n = 1 - \sqrt{1 - \left(\frac{b_n}{2}\right)^2} \tag{4}$$

with $A_0 = 2$ and $b_0 = 2$.

Now, by noticing that h_n can be expressed in terms of b_n, we can further simplify the relations (2) and (3). In effect,

$$\begin{aligned} b_n &= \sqrt{\left(\frac{b_{n-1}}{2}\right)^2 + \left(1 - \sqrt{1 - \left(\frac{b_{n-1}}{2}\right)^2}\right)^2} \\ &= \sqrt{2}\sqrt{1 - \sqrt{1 - \left(\frac{b_{n-1}}{2}\right)^2}} \end{aligned}$$

By eliminating the square roots signs from this last relation, one can check that

$$b_n^2 - \frac{b_n^4}{4} = \left(\frac{b_{n-1}}{2}\right)^2$$

Next,

$$\begin{aligned} A_n &= 2^n b_n h_n \\ &= 2^n b_n \left(1 - \sqrt{1 - \left(\frac{b_n}{2}\right)^2}\right) \\ &= 2^n \left(b_n - \sqrt{b_n^2 - \frac{b_n^4}{4}}\right) = 2^n \left(b_n - \frac{b_{n-1}}{2}\right) \end{aligned}$$

Finally, we can see that π can be expressed in the form of a telescoping series, that is

$$\begin{aligned} \pi &= \lim_{n \to +\infty} \left(\sum_{k=0}^{n} A_k\right) \\ &= \lim_{n \to +\infty} \left(A_0 + \sum_{k=1}^{n} 2^k \left(b_k - \frac{b_{k-1}}{2}\right)\right) \\ &= \lim_{n \to +\infty} \left(A_0 + \sum_{k=1}^{n} 2^k b_k - \sum_{k=0}^{n-1} 2^k b_k\right) \\ &= \lim_{n \to +\infty} \left(A_0 + 2^n b_n - 2^0 b_0\right) = \lim_{n \to +\infty} (2^n b_n) \end{aligned}$$

This completes the proof of Theorem 1. □

Remark 1. The above method is similar to the 2^k-gons methods to compute the area of a circle [2]. Also, the areas of the 2^k-gons are not computed as a whole but as the difference of two successive 2^k-gons. One can therefore use trigonometry and the formula of the area of a 2^k-gon to obtain the above result. However, here for the sake of simplicity we have preferred to use basic geometry to prove the above theorem and also to emphasize the geometric nature of the result.

We now present our next result:

Theorem 2. *Let $(c_n)_{n\geq 0}$ be a sequence defined by:*

$$c_n = \frac{4\left(\sqrt{1+\left(\frac{c_{n-1}}{2}\right)^2}-1\right)}{c_{n-1}} \tag{cn}$$

with $c_0 = 2$. Then,

$$\pi = \lim_{n\to+\infty} 2^{n+1} c_n. \tag{5}$$

Proof. The proof of this theorem is similar to previous one but instead of inscribing a square into the circle, we circumscribe a square outside the circle as depicted in Figure 4.

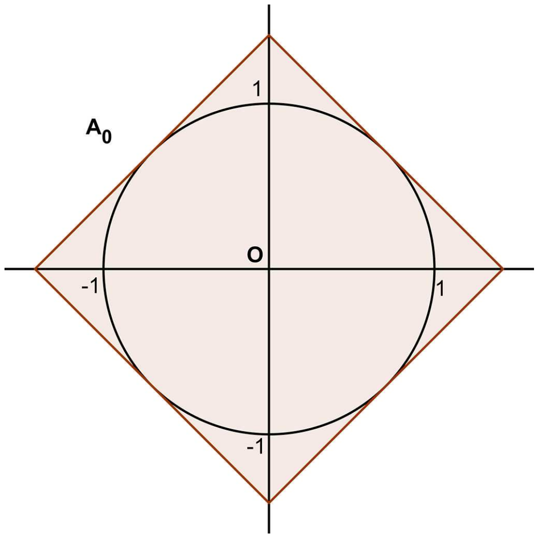

Figure 4: 1st approximation - A_0

The area of the square will be denoted as A_0. The side of the square has a length of 2, so $A_0 = 2 \times 2 = 4$. A_0 is a very rough approximation of the area of the circle ($\pi \simeq A_0$).

Next, to make a better approximation of the area of the circle we subtract the areas of the four shaded triangles, A_1, from A_0 as depicted in Figure 5. Similarly we shall proceed to remove the area of the eight shaded triangles, denoted by A_2, as shown in Figure 6.

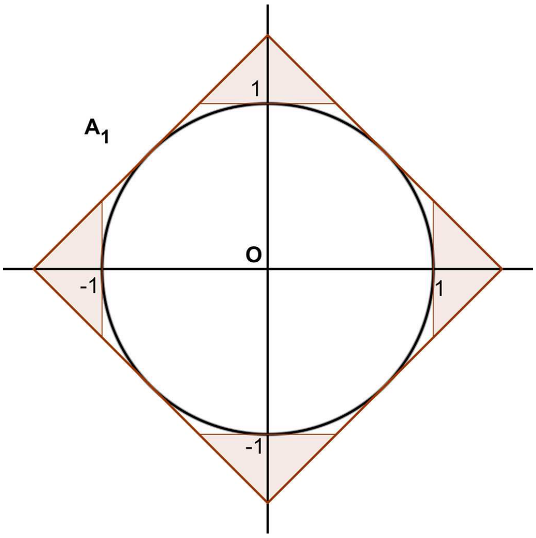

Figure 5: 2nd approximation: $A_0 - A_1$

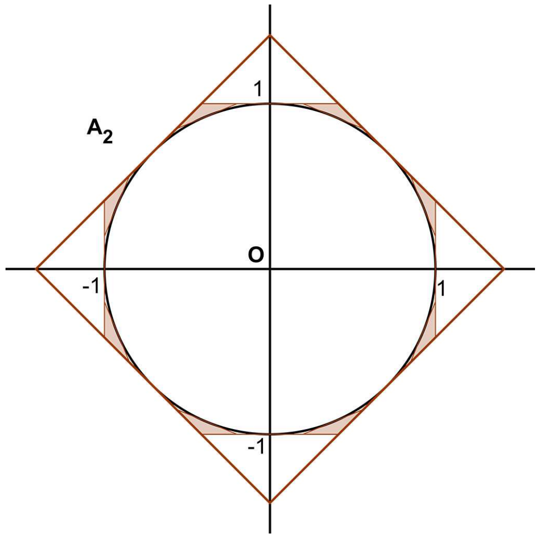

Figure 6: 3rd approximation: $A_0 - A_1 - A_2$

Finally, by repeating this operation infinitely, we obtain: $\pi = A_0 - \sum_{n=1}^{+\infty} A_n$.

Now, we need to find the expression for A_n for all $n \in \mathbb{N}$. For this, let us go back to the second approximation and concentrate only on one triangle and determine its area as shown in Figure 7.

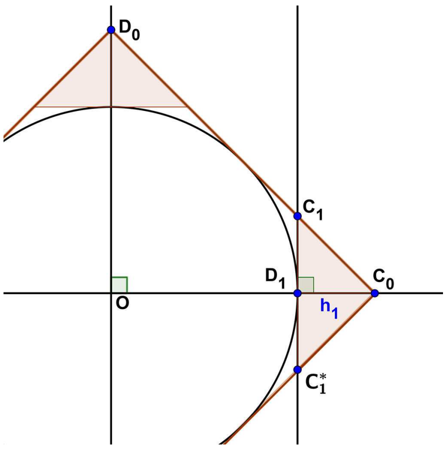

Figure 7: Determination of A_1

Let $c_1 = |C_1C_1^*|$ be the length of the base of the triangle $C_1C_0C_1^*$, and $h_1 = |D_1C_0|$ be the length of the height of the same triangle. It is easy to see that:

$$h_1 = |D_1C_0| = |OC_0| - |OD_1| = \sqrt{2} - 1.$$

Since the triangles OC_0D_0 and $D_1C_0C_1$ are similar, we have:

$$\frac{h_1}{1 + h_1} = \frac{c_1/2}{1 + h_1}$$

That is,

$$c_1 = 2h_1 = 2\sqrt{2} - 2.$$

Consequently,

$$A_1 = 4 \times \frac{c_1}{2} \times h_1 = 12 - 8\sqrt{2} \simeq 0.6863$$

We shall use the same method to find A_2 (Figure 8).

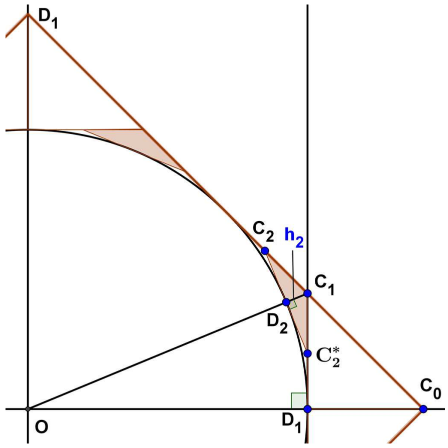

Figure 8: Determination of A_2

Let $c_2 = |C_2C_2^*|$ be the length of the base of the triangle $C_2C_1C_2^*$ and $h_2 = |D_2C_1|$ be the height of the same triangle. By using Pythagoras' theorem on the right angle triangle OC_1D_1, we can see that

$$|OC_1|^2 = |C_1D_1|^2 + |OD_1|^2,$$

so that

$$h_2 = \sqrt{1 + \left(\frac{c_1}{2}\right)^2} - 1.$$

Since OC_1D_1 and $D_2C_1C_2^*$ are two right angle and similar triangles, we have

$$\frac{|OD_1|}{|D_2C_2^*|} = \frac{|C_1D_1|}{|D_2C_1|}$$
$$\frac{1}{c_2/2} = \frac{c_1/2}{h_2}$$

That is

$$c_2 = \frac{4h_2}{c_1}$$

Consequently,

$$A_2 = 2^3 \times \frac{c_2}{2} \times h_2 \simeq 0.1311$$

We can now generalise the above for all $n \in \mathbb{N}$.

$$c_0 = 2 \tag{6}$$

$$h_n = \sqrt{1 + \left(\frac{c_{n-1}}{2}\right)^2} - 1 \tag{7}$$

$$c_n = \frac{4h_n}{c_{n-1}} \tag{8}$$

$$A_n = 2^n c_n h_n \tag{9}$$

$$\pi = 4 - \sum_{n=1}^{+\infty} A_n. \tag{10}$$

We can further simplify the above relations. First, from (8) and (7), we can see that

$$\begin{aligned} c_n h_n &= \frac{4h_n^2}{c_{n-1}} \\ &= \frac{4\left(\sqrt{1+\left(\frac{c_{n-1}}{2}\right)^2}-1\right)^2}{c_{n-1}} \\ &= c_{n-1} - \frac{8h_n}{c_{n-1}} = c_{n-1} - 2c_n. \end{aligned}$$

As a result,

$$A_n = 2^n c_n h_n = 2^n\left(c_{n-1} - 2c_n\right).$$

Consequently, here again, π is expressed in terms of a telescoping series

$$\pi = 4 - \sum_{n=1}^{+\infty} 2^n\left(c_{n-1} - 2c_n\right).$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{n} A_k &= \sum_{k=1}^{n}\left[2^k\left(c_{k-1} - 2c_k\right)\right] \\ &= 2c_0 - 2^{n+1}c_n = 4 - 2^{n+1}c_n. \end{aligned}$$

Hence,

$$\begin{aligned} \pi &= \lim_{n\to+\infty}\left(4 - \sum_{k=1}^{n} A_k\right) \\ &= \lim_{n\to+\infty}\left(4 - \left(4 - 2^{n+1}c_n\right)\right) = \lim_{n\to+\infty}\left(2^{n+1}c_n\right). \end{aligned}$$

This completes the proof of Theorem 2. $\square$

Now as a corollary, we can state the following

Corollary 1. *Let $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 0}$ be two sequences defined by:*

$$b_n = \sqrt{2}\sqrt{1-\sqrt{1-\left(\frac{b_{n-1}}{2}\right)^2}} \tag{11}$$

and

$$c_n = \frac{4\left(\sqrt{1+\left(\frac{c_{n-1}}{2}\right)^2}-1\right)}{c_{n-1}} \tag{12}$$

with $b_0 = c_0 = 2$. *Then,*

$$\lim_{n\to+\infty} 2^n b_n \leq \pi \leq \lim_{n\to+\infty} 2^{n+1} c_n. \tag{13}$$

We now present some numerical results, in the table below, that were computed using Matlab up to 12 decimal places.

Iteration step (n)	$2^n b_n$	$2^n c_n$
1	2.828427124746	3.313708498985
2	3.061467458921	3.182597878075
3	3.121445152258	3.151724907429
4	3.136548490546	3.144118385246
5	3.140331156955	3.142223629942
6	3.141277250933	3.141750369169
7	3.141513801144	3.141632080703
8	3.141572940367	3.141602510257
9	3.141587725277	3.141595117750
10	3.141591421511	3.141593269629

Table 1.

It can be seen that after iteration 5, we start obtain reasonable value for π. In fact, more accurate approximation for π is obtained for $n = 17$, where $2^{17} b_{17} = 3.141592653515$ and $2^{17} c_{17} = 3.141592653627$.

3 Conclusions

In this work, we have proposed a geometrical method to compute π based on Vieta's approach. The proposed result is not intended for computing π with very high accuracy; although it gives reasonable approximation compared to other geometrical methods such that of Archimedes and Vieta. The results obtained are only intended to fill some gaps in the existing geometrical methods that is available to compute π.

References

[1] Arndt, J., & Haenel, C. (2006) *Pi Unleashed*. Springer-Verlag.

[2] Borwein, J. M., Borwein, P. B., & Bailey, D. H. (1989) Ramanujan, Modular Equations, and Approximations to Pi or How to Compute One Billion Digits of Pi. *The American Mathematical Monthly*, 96(3), 201–219.

[3] Delahaye, J.-P. (1997) *Le fascinant nombre Pi*. Éditions Belin/Pour la science.

[4] Guillera Gomez, J. (2007) Historia de las formulas y algoritmos para π. *Las Gaceta de la Real Sociedad Matematica Espanola*, 10(1), 159–178.

[5] Guillera, J. (2006) A new method to obtain series for $1/\pi$ and $1/\,2\pi$, *Exp. Math.*, 15, 83–89.

[6] Osler, T. J. (1999) The united Vieta's and Wallis's products for π. *American Mathematical Monthly*, 106, 774–776.

[7] Salamin, E. (1976) Computation of using arithmetic-geometric mean. *Math. Comp.*, 30, 565–570.

[8] Osler, T. J. (2007) Vieta-like products of nested radicals with Fibonacci and Lucas numbers. *The Fibonacci Quarterly.* 52(1), 202–204.