Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 22, 2016, No. 4, 56–61

Right circulant determinant sequences with Jacobsthal and Jacobsthal–Lucas Numbers

Aldous Cesar F. Bueno

Philippine Science High School, Central Luzon Campus
Lily Hill, Clark Freeport Zone, Pampanga, Philippines
e-mail: afbueno@clc.pshs.edu.ph

Received: 30 September 2015 **Accepted:** 30 October 2016

Abstract: We study two right circulant determinant sequences. The first sequence makes use of Jacobsthal numbers of the form J_{s+t} while the other makes use of Jacobsthal–Lucas numbers of the form K_{s+t}, where $s,\ t \in \mathbb{Z}$ and $s \neq t$. We also give some open problems.

Keywords: Determinants sequence, Jacobsthal numbers, Jacobsthal–Lucas numbers.
AMS Classification: 15B05.

1 Introduction

In the study of right circulant matrices, it has been a tradition to use number sequences as entries. From these formed matrices, the eigenvalues, determinants and norms are computed and then analyzed. Some works on these kinds of venture on right circulant matrices are that of Bahsi [1], Bozkurt [2–4], Bueno [5–10], Civciv and Turkmen [11], Nalli and Sen [14], Shen and Cen [15] and Yalciner [16].

In this paper, we form sequences from the determinants of right circulant matrices. This kind of sequence was introduced by Majumdar in [13] and he made use of arithmetic sequence and natural numbers. For this instance, we will be using Jacobsthal and Jacobsthal–Lucas numbers. The sequences that we will be dealing with are as follows

$$\{A_n\} = \left\{ |J_s|, \begin{vmatrix} J_s & J_{s+t} \\ J_{s+t} & J_s \end{vmatrix}, \begin{vmatrix} J_s & J_{s+t} & J_{s+2t} \\ J_{s+2t} & J_s & J_{s+t} \\ J_{s+t} & J_{s+2t} & J_s \end{vmatrix}, \dots \right\}, \tag{1}$$

$$\{B_n\} = \left\{ |K_s|, \begin{vmatrix} K_s & K_{s+t} \\ K_{s+t} & K_s \end{vmatrix}, \begin{vmatrix} K_s & K_{s+t} & K_{s+2t} \\ K_{s+2t} & K_s & K_{s+t} \\ K_{s+t} & K_{s+2t} & K_s \end{vmatrix}, \ldots \right\}, \tag{2}$$

where $J_k = \frac{2^k-(-1)^k}{3}$, the k^{th} Jacobsthal number, $K_k = 2^k + (-1)^k$, the k^{th} Jacobsthal–Lucas number, $s,\ t \in \mathbb{Z}$ and $s \neq t$.

Our goal is to determine their n^{th} terms, bounds for their sums and provide some results on divisibility.

2 Main results

Theorem 2.1.

$$A_n = \frac{(J_s - J_{s+nt})^n - [f(s,t) - g(s,t,n)]^n}{1 - K_{nt} + (-2)^{nt}}, \tag{3}$$

where $f(s,t) = \frac{2^s\cdot(-1)^t-(-1)^s\cdot 2^t}{3}$ *and* $g(s,t,n) = \frac{2^{s+nt}\cdot(-1)^t-(-1)^{s+nt}\cdot 2^t}{3}$.

Proof. We first compute for the eigenvalues of the matrix

$$C_n(\vec{J}) = \begin{pmatrix} J_s & J_{s+t} & J_{s+2t} & \cdots & J_{s+(n-2)t} & J_{s+(n-1)t} \\ J_{s+(n-1)t} & J_s & J_{s+t} & \cdots & J_{s+(n-3)t} & J_{s+(n-2)t} \\ J_{s+(n-2)t} & J_{s+(n-1)t} & J_s & \cdots & J_{s+(n-4)t} & J_{s+(n-3)t} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J_{s+2t} & J_{s+3t} & J_{s+4t} & \cdots & J_s & J_{s+t} \\ J_{s+t} & J_{s+2t} & J_{s+3t} & \cdots & J_{s+(n-1)t} & J_s \end{pmatrix}.$$

The eigenavlues are given by

$$\begin{aligned} \alpha_m &= \sum_{k=0}^{n-1} J_{s+kt}\omega^{-mk} \\ &= \frac{1}{3}\sum_{k=0}^{n-1} \left[2^{s+kt} - (-1)^{s+kt}\right]\omega^{-mk} \end{aligned}$$

$m = 0\ 1,\ \ldots n-1$ and $\omega = e^{\pi i/n}$.

Simplfying these expressions, we will end up with

$$\alpha_m = \frac{J_s - J_{s+nt} - [f(s,t) - g(s,t,n)]\omega^{-m}}{(1-2^t\omega^{-m})(1-(-1)^t\omega^{-m})},$$

where $f(s,t) = \frac{2^s\cdot(-1)^t-(-1)^s\cdot 2^t}{3}$ and $g(s,t,n) = \frac{2^{s+nt}\cdot(-1)^t-(-1)^{s+nt}\cdot 2^t}{3}$.

Note that A_n is just the determinant of $C_n(\vec{J})$, so it's just the product of all its eigenvalues. Hence

$$\begin{aligned}A_n &= \prod_{m=0}^{n-1} \alpha_m \\ &= \prod_{m=0}^{n-1} \left[\frac{J_s - J_{s+nt} - [f(s,t) - g(s,t,n)]\omega^{-m}}{(1-2^t\omega^{-m})(1-(-1)^t\omega^{-m})}\right] \\ &= \frac{(J_s - J_{s+nt})^n - [f(s,t) - g(s,t,n)]^n}{(1-2^{nt})(1-(-1)^{nt})}.\end{aligned}$$

Since $\prod_{m=0}^{n-1}(x - y\omega^{-m}) = x^n - y^n$, then

$$A_n = \frac{(J_s - J_{s+nt})^n - [f(s,t) - g(s,t,n)]^n}{1 - K_{nt} + (-2)^{nt}}$$

which is as desired. □

Theorem 2.2.

$$B_n = \frac{(K_s - K_{s+nt})^n - [p(s,t) - q(s,t,n)]^n}{1 - K_{nt} + (-2)^{nt}}, \tag{4}$$

where $p(s,t) = 2^s \cdot (-1)^t + (-1)^s \cdot 2^t$ *and* $q(s,t,n) = 2^{s+nt} \cdot (-1)^t + (-1)^{s+nt} \cdot 2^t$.

Proof is the same as as that of Theorem 2.1.

Corollary 2.3.

$$(J_s - J_{s+nt})^n - [f(s,t) - g(s,t,n)]^n$$

and

$$(K_s - K_{s+nt})^n - [p(s,t) - q(s,t,n)]^n$$

are both divisible by

$$1 - K_{nt} + (-2)^{nt}.$$

Proof. Note that A_n are B_n are determinants of integer matrices, and the determinant of integer matrices are integers. This means that A_n and B_n are also integers, so it immediately follows that $1 - K_{nt} + (-2)^{nt}$ divides $(J_s - J_{s+nt})^n - [f(s,t) - g(s,t,n)]^n$ and $(K_s - K_{s+nt})^n - [p(s,t) - q(s,t,n)]^n$. □

Theorem 2.4. *Let* $\epsilon = \min\{|1 - K_{mt} + (-2)^{mt}|\}$, $M(m) = \max\left\{\begin{pmatrix} n \\ m \end{pmatrix}\right\}$, $\Omega = \max\{|f(s,t) - g(s,t,n)|\}$ *and* $\Theta = \max\{|p(s,t) - q(s,t,n)|\}$. *Then*

$$\sum_{m=1}^{n} |A_m| \le \frac{1}{\epsilon}\left[\frac{1-\Omega^n}{1-\Omega} + \sum_{m=1}^{n}\sum_{k=1}^{m} M(m)\left|J_s^m \cdot J_{s+mt}^{m-k}\right|\right] \tag{5}$$

$$\sum_{m=1}^{n} |B_m| \le \frac{1}{\epsilon}\left[\frac{1-\Theta^n}{1-\Theta} + \sum_{m=1}^{n}\sum_{k=1}^{m} M(m)\left|K_s^m \cdot K_{s+mt}^{m-k}\right|\right] \tag{6}$$

Proof.

$$\begin{aligned}\sum_{m=1}^{n}|A_m| &= \sum_{m=1}^{n}\left|\frac{(J_s-J_{s+mt})^m-[f(s,t)-g(s,t,m)]^m}{1-K_{mt}+(-2)^{mt}}\right| \\ &\leq \frac{1}{\epsilon}\sum_{m=1}^{n}|(J_s-J_{s+mt})^m-\Omega^m|] \\ &\leq \frac{1}{\epsilon}\sum_{m=1}^{n}[|(J_s-J_{s+mt})^m|+\Omega^m]\end{aligned}$$

where $\epsilon=\min\{|1-K_{mt}+(-2)^{mt}|\}$ and $\Omega=\max\{|f(s,t)-g(s,t,n)|\}$.
Continuing we have

$$\begin{aligned}\sum_{m=1}^{n}|A_m| &\leq \frac{1}{\epsilon}\left[\sum_{m=1}^{n}|(J_s-J_{s+mt})^m|+\sum_{m=1}^{n}\Omega^m\right] \\ &\leq \frac{1}{\epsilon}\left[\frac{1-\Omega^n}{1-\Omega}+\sum_{m=1}^{n}\sum_{k=1}^{m}\begin{pmatrix} n \\ m\end{pmatrix}\left|J_s^m\cdot J_{s+mt}^{n-m}\right|\right] \\ &\leq \frac{1}{\epsilon}\left[\frac{1-\Omega^n}{1-\Omega}+\sum_{m=1}^{n}\sum_{k=1}^{m}M(m)\left|J_s^m\cdot J_{s+mt}^{m-k}\right|\right],\end{aligned}$$

where $M(m)=\max\left\{\begin{pmatrix} n \\ m\end{pmatrix}\right\}$.

Similarly, we have

$$\sum_{m=1}^{n}|B_m|\leq\frac{1}{\epsilon}\left[\frac{1-\Theta^n}{1-\Theta}+\sum_{m=1}^{n}\sum_{k=1}^{m}M(m)\left|K_s^m\cdot K_{s+mt}^{m-k}\right|\right],$$

where $\Theta=\max\{|p(s,t)-q(s,t,n)|\}$. □

3 Conclusions

First, we have established that the n^{th} terms of $\{A_n\}$ and $\{B_n\}$ are always integers and as a consequence we have divisibilities involving $1-K_{nt}+(-2)^{nt}$.

Lastly, using simple triangle inequality and cocncepts of extremum values, we have found bounds for $\sum_{m=1}^{n}|A_m|$ and $\sum_{m=1}^{n}|B_m|$.

4 Open problems

Here are some problems for further study

- Identities that can be derived from the n^{th} terms of $\{A_n\}$ and $\{B_n\}$, that is, Laplacian expansion versus formulas (3) and (4)

- Periodicity on the the n^{th} terms of $\{A_n\}$ and $\{B_n\}$
- Sharper bounds for $\sum_{m=1}^{n} |A_m|$ and $\sum_{m=1}^{n} |B_m|$
- Explicit formula for $\sum_{m=1}^{n} A_m$ and $\sum_{m=1}^{n} B_m$
- Investigate the following determinant sequences

$$\{U_n\} = \left\{ |J_s|, \begin{vmatrix} J_s & J_{st} \\ J_{st} & J_s \end{vmatrix}, \begin{vmatrix} J_s & J_{st} & J_{st^2} \\ J_{st^2} & J_s & J_{st} \\ J_{st} & J_{st^2} & J_s \end{vmatrix}, \ldots \right\},$$

$$\{V_n\} = \left\{ |K_s|, \begin{vmatrix} K_s & K_{st} \\ K_{st} & K_s \end{vmatrix}, \begin{vmatrix} K_s & K_{st} & K_{st^2} \\ K_{st^2} & K_s & K_{st} \\ K_{st} & K_{st^2} & K_s \end{vmatrix}, \ldots \right\},$$

where $s, t \in \mathbb{Z}/\left\{0\right\}$ and s and t are not both equal to 1

References

[1] Bahsi, M. & Solak, S.(2010) On the Circulant Matrices with Arithmetic Sequence, *International Journal of Contemporary Matematical. Sciences*, 5(25), 1213–1222.

[2] Bozkurt, D. (2012) On the Determinants and Inverses of Circulant Matrices with a General Number Sequence, *arXiv: 1202.1068v1 [math.NA] 6 Feb 2012.*

[3] Bozkurt, D. (2012) On the Determinants and Inverses of Circulant Matrices with Jacobsthal and Lucas–Jacobsthal Numbers, *arXiv: 1201.6058v1 [math.NA] 29 Jan 2012.*

[4] Bozkurt, D. (2012) On the Determinants and Inverses of Circulant Matrices with Pell and Pell–Lucas Numbers, *arXiv: 1201.6061v1 [math.NA] 29 Jan 2012.*

[5] Bueno, A.C.F. (2012) On Right Circulant Matrices with Geometric Progression, *International Journal of Applied Mathematical Research*, 1(4), 593–603.

[6] Bueno, A.C.F. (2012) Right Circulant Matrices with Fibonacci Sequence, *International Journal of Mathematics and Scientific Computing*, 2(2), 9–10.

[7] Bueno, A.C.F. (2013) Right Circulant Matrices with Jacobsthal Sequence, *International Journal of Advanced Mathematical Sciences*, 1(2), 87–90.

[8] Bueno, A.C.F. (2013) Generalized Right Circulant Matrices with Geometric Sequence, *International Journal of Mathematics and Scientific Computing*, 3(1), 17–18

[9] Bueno, A.C.F. (2014) On the Eigenvalues and the Determinant of the Right Circulant Matrices with Pell and Pell–Lucas Numbers, *International Journal of Mathematics and Scientific Computing*, 4(1), 19–20.

[10] Bueno, A.C.F. (2014) Right Circulant Matrices with Sum of the Terms of Two Geometric Sequences, *International Journal of Mathematics and Scientific Computing*, 4(2), 51–52.

[11] Civciv, H. & Turkmen, R. Notes on Norms of Circulant Matrices with Lucas Numbers, *International Journal of Information and System Sciences*, 4(1), 142–147.

[12] Gray, R. Toeplitz and Circulant Matrices: A Review, `http://ee.stanford.edu/~gray/toeplitz.pdf`

[13] Majumdar, A.A.K. Wandering in the World of Smarandache Numbers, `http://fs.gallup.unm.edu/Majumdar.pdf`

[14] Nalli, A. & Sen, M. (2010) On the Norms of Circulant Matrices with Generalized Fibonacci Numbers, *Selcuk Journal of Applied Mathematics*, 11(1), 107–116.

[15] Shen, S. & Cen, J. (2011) On the Norms of Circulant Matrices with the (k, h)-Fibonacci and (k, h)-Lucas Numbers, International Journal of Contemporary Mathematics and Sciences, 6(19), 887–894.

[16] Yalciner, A. (2008) Spectral Norms of Some Special Circulant Matrices, International Journal of Contemporary Mathematics and Sciences, 3(35), 1733–1738.