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Figurate numbers in the modular ring Z_4

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Abstract: The sums of odd integers in classes $\bar{1}_4, \bar{3}_4 \subset Z_4$, a modular ring, show clear distinctions between the two classes. In particular, the sum for class $\bar{1}_4$ is related to the Golden Ratio family of sequences, and in this class when the position of an odd integer is a prime number, then the sum always has a factor of 6. Sums of the primes in these classes can be primes but the structures are quite different, and no sums of odd integers in general are primes. The sums are related to the sequences of triangular numbers and hexagonal numbers.

Keywords: Prime numbers, Composite numbers, Right-end-digits, Modular rings, Generalized Golden Ratio, Generalized Fibonacci numbers, Triangular numbers, Hexagonal numbers.
AMS Classification: 11B39, 11B50.

1 Introduction

It is well-known that the sum of odd integers equals a square [5, 6]. Here we consider the sums of odd integers in the modular ring Z_4 (Table 1) where the odd integers fall in classes $\bar{1}_4, \bar{3}_4 \subset Z_4$.

Row $r_i \downarrow$	**Class** $i \rightarrow$	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$	**Comments**
0		0	1	2	3	• $N = 4r_i + i$
1		4	5	6	7	• even $\bar{0}_4, \bar{2}_4$
2		8	9	10	11	• $(N^n, N^{2n}) \in \bar{0}_4$
3		12	13	14	15	• odd $\bar{1}_4, \bar{3}_4$; $N^{2n} \in \bar{1}_4$

Table 1. Classes and rows for Z_4

It has recently been shown that the integers associated with the generalized Golden Ratio φ_a are class specific. Only those in $\bar{1}_4$ equal a in the function $\frac{1}{2}\left(1+\sqrt{a}\right)$, a zero of the quadratic polynomial associated with the generalized Golden Ratio and generalized Fibonacci numbers

$$x^2 - x - r_1$$

in which r_1 is the row of integers in $\bar{1}_4$. It is the purpose of this paper to demonstrate new insights among known results by comparing Fibonacci numbers and figurate numbers [1] as primes and sums $S_m \in \bar{1}_4$ and $S_n \in \bar{3}_4$.

2 Integer sums in class $\bar{1}_4$

Integers, $N_n \in \bar{1}_4$ have the form $4r_1 + 1$, the triangular numbers, T_n, [5]

$$T_n = \sum_{j=1}^{n} j,$$

which has the well-known general term

$$T_n = \tfrac{1}{2}n(n+1), \qquad (2.1)$$

and when $r_1 = 0$, $N = 1$, so that

$$S_n = 4T_n + n + 1 = 2n^2 + 3n + 1, \qquad (2.2)$$

or, the more familiar

$$S_{n-1} = n(2n-1),$$

as in Table 2 for $n = 0,1,2,\ldots 27$. The S_n are the hexagonal numbers [2,3].

N_n	n	S_n	Factors	N_n	n	S_n	Factors
1	0	1	1	57	14	435	3,5,29
5	1	6	2,3	61	15	496	2,2,2,2,31
9	2	15	3,5	65	16	561	3,11,17
13	3	28	4,7	69	17	630	2,3,3,5,7
17	4	45	3,5	73	18	703	19,37
21	5	66	2,3,11	77	19	780	2,2,3,5,13
25	6	91	7,13	81	20	861	3,7,41
29	7	120	2,2,3,5	85	21	946	2,11,43
33	8	153	3,3,17	89	22	1035	3,3,5,23
37	9	190	2,5,19	93	23	1128	2,2,2,3,47
41	10	231	3,7,11	97	24	1225	5,5,7,7
45	11	276	2,2,3,23	101	25	1326	2,3,13,17
49	12	325	5,5,13	105	26	1431	3,3,3,53
53	13	378	2,3,3,7	109	27	1540	4,5,7,11

Table 2: S_n

When n is prime greater than 3, $6|S_n$ and the parity sequence is oeeooee... (o: odd; e: even), but no S_n equals a prime and the factors follow regular functions (Table 3).

Factor	***n***
3	$1 + 3t$ $2 + 3t$
5	$2 + 5t$ $4 + 5t$
7	$3 + 7t$ $6 + 7t$
11	$5 + 11t$ $10 + 11t$

Table 3: S_n factor functions $t = 0,1,2,3,...$

It is clear that factors, u, of S_n occur when

$$n = b + ut,\ t = 0, 1, 2, 3, ... \tag{2.3}$$

in which $b = ½(u – 1)$ or $(u – 1)$. Furthermore, the class structure of S_n is

$$\bar{1}_4\bar{2}_4\bar{3}_4\bar{0}_4\bar{1}_4\bar{2}_4\bar{3}_3\bar{0}_4..$$

so that $S_n \in \bar{1}_4\bar{2}_4\bar{3}_4\bar{0}_4$ when

$$n = c + 4t$$

where $c = 0, 1, 2, 3, 4$, for each class respectively.

It can also be observed from Table 2 that $n = r_1$, so that from Equation (2.2)

$$r_1 = \frac{-3+\sqrt{1+8S_n}}{4}. \tag{2.4}$$

so that there is a link through r_1 among the Golden Ratio family φ_n [4], generalized Fibonacci numbers [7] and S_n. These generalized Fibonacci sequences satisfy the second order recurrence relation

$$F_{n+1} = F_n + r_1 F_{n-1} \tag{2.5}$$

with the generalized Golden ratio

$$\varphi_a = \frac{1+\sqrt{a}}{2},\ a \in \bar{1}_4. \tag{2.6}$$

Thus since $a = 4r_1 + 1$,

$$a = \sqrt{1+8S_n} - 2 \tag{2.7}$$

from Equation (2.4). Hence

$$(2\varphi_a - 1)^2 = \sqrt{1+8S_n} - 2. \tag{2.8}$$

For example, with $S_{17} = 630$, we have

$$1 + 8S_n = 5041 = 71^2,$$

so that

$$\left(2\varphi_a - 1\right)^2 = 69$$

and

$$\varphi_{69} = \tfrac{1}{2}\left(1 + \sqrt{69}\right).$$

3 Integer sums in class $\bar{3}_4$

Integers, $N_n \in \bar{3}_4$ have the form $4r_3 + 3$, so that for this class

$$S_m = 4T_m + 3m + 3 \qquad (3.1)$$

$$= 2m^2 + 5m + 3 \qquad (3.2)$$

or, the more familiar

$$S_{m-1} = m(2m+1).$$

as in Table 4 for m = 0,1,2,...231. The S_m are the hexagonal numbers of the second kind [2, 3].

N_m	m	S_m	Factors	N_m	m	S_m	Factors
3	0	3	3	71	17	666	2,3,37
7	1	10	2,5	75	18	741	3,13,19
11	2	21	3,7	79	19	820	2,5,41
15	3	36	2,2,3,3	83	20	903	3,7,43
19	4	55	5,11	87	21	990	2,3,5,11
23	5	78	2,3,13	91	22	1081	23,47
27	6	105	3,5,7	95	23	1176	2,2,2,3,7,7
31	7	136	2,2,2,17	99	24	1275	3,5,17
35	8	171	3,3,19	103	25	1378	2,13,53
39	9	210	2,3,5,7	107	26	1485	3,5,11
43	10	253	11,23	111	27	1596	2,2,3,7,19
47	11	300	2,2,3,5,5	115	28	1711	29,59
51	12	351	3,3,3,13	119	29	1830	2,3,5,61
55	13	406	2,7,29	123	30	1953	3,3,7,31
59	14	465	3,5,31	127	31	2080	2,2,2,2,2,5,13
63	15	528	2,2,2,2,3,11	131	32	2211	3,11,67
67	16	595	5,7,17	135	33	2346	2,3,17,23

Table 4: S_m

As with Class $\bar{1}_4$, the S_m sequence contains no primes, and the factors, v, follow regular functions (Table 5). It is clear there that the factors, v, of S_m occur when

$$m = b + vt,\ t = 0, 1, 2, 3, \ldots \qquad (3.3)$$

in which $b = ½(u - 3)$ or $(u - 1)$.

Factor	*m*
3	$3t$ $2 + 3t$
5	$1 + 5t$ $4 + 5t$
7	$2 + 7t$ $6 + 7t$
11	$4 + 11t$ $10 + 11t$

Table 5: S_m factor functions $t = 0, 1, 2, 3, ...$

4 The sums of primes

Some examples of the partial sums $S_n(p), S_m(p)$ for primes *p* are shown in Tables 6 and 7:

- In Table 6, for class $\bar{1}_4$, 20% of the partial sums on display are primes, and, except for $n = 9$, all *n* are primes when S_n is prime.
- In Table 7, for class $\bar{3}_4$, 19% of the partial sums on display are primes, but are not specific to prime *m*.

n	*p*	$S_n(p)$	*n*	*p*	$S_n(p)$
1	5	5 (p)	23	229	2311(p)
2	13	18	24	233	2544
3	17	35	25	241	2785
4	29	64	26	257	3042
5	37	101(p)	27	269	3311
6	41	142	28	277	3588
7	53	195	29	281	3869
8	57	252	30	293	4162
9	61	313(p)	31	313	4475
10	73	386	32	317	4792
11	89	475	33	337	5129
12	97	572	34	349	5478
13	101	673(p)	35	353	5831
14	109	782	36	373	6204
15	113	895	37	389	6593
16	137	1032	38	397	6990
17	149	1181(p)	39	401	7391
18	157	1338	40	409	7800
19	173	1511(p)	41	421	8221(p)
20	181	1692	42	433	8654
21	193	1885	43	449	9103(p)
22	197	2082	44	457	9560

Table 6: Primes $p \in \bar{1}_4$

n	p	$S_m(p)$	n	p	$S_m(p)$
1	3	3(p)	23	191	2001
2	7	10	24	199	2200
3	11	21	25	211	2411(p)
4	19	40	26	223	2634
5	23	63	27	227	2861(p)
6	31	94	28	239	3100
7	43	137(p)	29	263	3363
8	47	184	30	271	3634
9	59	243	31	283	3917(p)
10	67	310	32	307	4224
11	71	381	33	311	4535
12	79	460	34	331	4866
13	83	543	35	347	5213
14	103	646	36	359	5572
15	107	753	37	367	5939(p)
16	127	880	38	379	6318
17	131	1011	39	383	6701(p)
18	139	1150	40	419	7120
19	151	1301(p)	41	431	7551
20	163	1464	42	439	7990
21	167	1631	43	443	8433
22	179	1810	44	463	8896

Table 7: Primes $p \in \bar{3}_4$

When a prime sum occurs for a prime in these classes that prime usually has an odd row for $\bar{1}_4$ but an even row for $\bar{3}_4$, and the right-end-digits (REDs) are commonly, but not always, 1 or 3 in Tables 6 and 7. Moreover, the RED and Class structures are quite distinct (Tables 8 and 9).

Class	Odd Sum	Even Sum
$\bar{1}_4$	55153535115151959131131	8422622282242822840040
$\bar{3}_4$	31373133111111375391133	0044006004004044628006

Table 8: RED structure of prime sums

Class	Classes
$\bar{1}_4$	$\bar{1}_4\bar{2}_4\bar{3}_4\bar{0}_4\bar{1}_4\bar{2}_4\bar{3}_4\bar{0}_4\ldots$
$\bar{3}_4$	$\bar{3}_4\bar{2}_4\bar{1}_4\bar{0}_4\bar{3}_4\bar{2}_4\bar{1}_4\bar{0}_4\ldots$

Table 9: Class structure of prime sums

The differences for the two classes are consistent with the differences for the Golden Ratio family [4] and for the sum of squares [6].

Further investigation for the interested reader could be a study of the polynomials associated with these sequences, particularly in the context of their modular properties.

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