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Identities for balancing numbers using generating function and some new congruence relations

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Abstract: It is well-known that the balancing numbers are the square roots of the triangular numbers and are the solutions of the Diophantine equation $1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+r)$, where r is the balancer corresponding to the balancing number n. Thus if n is a balancing number, then $8n^2 + 1$ is a perfect square and its positive square root is called a Lucas-balancing number. The goal of this paper is to establish some new identities of these numbers.
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1 Introduction

A. Behera et.al [2] introduced the sequence of balancing numbers as follows. A positive integer n is called a balancing number with balancer r if it is the solution of Diophantine equation

$$1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+r).$$

The balancing numbers though obtained from a simple Diophantine equation, are very useful for the computation of square triangular numbers. An important result about balancing numbers is that, B is balancing number if and only if B^2 is a triangular number i.e. $8B^2+1$ is a perfect square. For each balancing number B, $C = \sqrt{8B^2+1}$ is called a Lucas-balancing number [9, 10]. First four balancing numbers are 1, 6, 35 and 204 with balancers 0, 2, 14 and 84 respectively. Let B_n and C_n are respectively denoted by n^{th} balancing number and n^{th} Lucas-balancing number. The balancing numbers satisfy the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1}; \;\; n \geq 2, \tag{1}$$

with $B_1 = 1$ and $B_2 = 6$ [2] whereas, the Lucas-balancing numbers defined recursively by

$$C_{n+1} = 6C_n - C_{n-1}; \;\; n \geq 2, \tag{2}$$

with $C_1 = 3$ and $C_2 = 17$ [9]. Many useful identities involving balancing and Lucas-balancing numbers are available in the literature. One can go through [1, 3, 4, 5, 6, 7, 8, 11, 12]. There are many well-known relationships between balancing and Lucas-balancing numbers. Most of the relationships were established from the Binet's formulas

$$B_n = \frac{\lambda^n - \lambda^{-n}}{2\sqrt{8}}, \quad C_n = \frac{\lambda^n + \lambda^{-n}}{2}, \tag{3}$$

where $\lambda = 3 + \sqrt{8}$ and $\lambda^{-1} = 3 - \sqrt{8}$. It is well known that matrices are used to represent the Fibonacci numbers. Also these can be used to represent balancing numbers and their related sequences. In [13], Ray has introduced a second order balancing Q_B matrix whose entries are the first three balancing numbers 0, 1 and 6. He has also shown that the n^{th} power of the balancing matrix Q_B is given by

$$Q_B^n = \begin{pmatrix} B_{n+1} & -Bn \\ B_n & -B_{n-1} \end{pmatrix}.$$

This matrix representation turns out to be an elegant way of finding relationships between the balancing and Lucas-balancing numbers. Ray, in [16], has established some balancing and Lucas-balancing sums using matrix method. The observation $\det(Q_B) = 1 = \det(Q_B^n)$ at once gives the Cassini formula for the balancing numbers $B_k^2 - B_{k+1}B_{k-1} = 1$.

In this article, we establish some combinatorial properties of balancing numbers and then establish some new congruences relations for these numbers.

2 Some known properties for balancing numbers by matrix method

In this section, we recover some well known properties of balancing numbers by matrix method.

2.1 Binet's formula

Consider a pair of two consecutive vectors $(B_{n+1},\ B_n)$ from the balancing sequence

$$\ldots, B_n, B_{n+1}, B_{n+2}, B_{n+3}, \ldots$$

In fact, the recurrence relation can be described in the following manner

$$\begin{pmatrix} 6 & -1 \end{pmatrix} \begin{pmatrix} B_{n+1} \\ B_n \end{pmatrix} = 6B_{n+1} - B_n = B_{n+2}.$$

The two consecutive vectors can be written in a matrix form as

$$\begin{pmatrix} B_{n+2} & -B_{n+1} \\ B_{n+1} & -B_n \end{pmatrix},$$

so that for $n = 0$ we recover the balancing matrix $Q_B = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$.

Also, it can be observed that

$$Q_B^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} B_{n+1} \\ B_n \end{pmatrix}. \tag{4}$$

The power of a non-diagonal matrix is difficult to compute, therefore in order to use (4) to describe B_n, we must diagonalize Q_B as follows: Clearly $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$ are eigenvalues of the matrix Q_B. These roots satisfy the following relations

$$\lambda_1^2 = 6\lambda_1 - 1, \;\; \lambda_2^2 = 6\lambda_2 - 1,$$

$$\lambda_1\lambda_2 = 1, \; \lambda_1 + \lambda_2 = 6, \; \lambda_1 - \lambda_2 = 2\sqrt{8}.$$

Also it has been observed that the eigenvectors corresponding to the eigenvalues λ_1 and λ_2 are of the form $(\lambda_1, \; 1)^T$ and $(\lambda_2, \; 1)^T$ respectively. Putting these eigenvectors into a change of basis matrix P which is in the form

$$P = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}.$$

If $\widetilde{Q_B} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ denotes the diagonalization of the balancing matrix Q_B and since $\lambda_1 - \lambda_2 = 2\sqrt{8}$, we have

$$Q_B = P\widetilde{Q_B}P^{-1},$$

where $P^{-1} = \frac{1}{2\sqrt{8}} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$. It follows that

$$Q_B^n = P\widetilde{Q_B}^n P^{-1}.$$

Thus by (4), we have

$$\begin{aligned} \begin{pmatrix} B_{n+1} \\ B_n \end{pmatrix} &= \frac{1}{2\sqrt{8}} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{2\sqrt{8}} \begin{pmatrix} \lambda_1^{n+1} + \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{pmatrix}. \end{aligned}$$

Comparing the second row and first column element, we obtain

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2},$$

which is popularly known as Binet's formula for balancing numbers.

2.2 Generating function of balancing numbers by matrix method

Recall that the generating function $G(x)$ for any sequence $\{a_n\}$ is given by the formula

$$G(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Behera et.al established the generating function for balancing numbers in [2] as

$$g(s) = \frac{s}{1 - 6s + s^2}.$$

This result can also be obtained by matrix method. Consider the matrix

$$I - sQ_B = \begin{pmatrix} 1-6s & s \\ -s & 1 \end{pmatrix}, \tag{5}$$

where I be the identity matrix same order as Q_B. Since the determinant value of the matrix (5) is $1 - 6s + s^2$, its inverse will be

$$(I - sQ_B)^{-1} = \frac{1}{1 - 6s + s^2} \begin{pmatrix} 1 & -s \\ s & 1-6s \end{pmatrix}. \tag{6}$$

Therefore, if $g(s) = \sum_{k=0}^{\infty} s^k Q_B^k = (I - sQ_B)^{-1}$ be the generating function, then using (6) we can write

$$s^0 Q_B^0 + s^1 Q_B^1 + s^2 Q_B^2 + \ldots = \begin{pmatrix} \frac{1}{1-6s+s^2} & \frac{-s}{1-6s+s^2} \\ \frac{s}{1-6s+s^2} & \frac{1-6s}{1-6s+s^2} \end{pmatrix}.$$

Comparing the like coefficient from both sides, we get the generating function for balancing numbers as

$$B_0 + sB_1 + s^2 B_2 + \ldots = g(s) = \frac{s}{1 - 6s + s^2}. \tag{7}$$

If we choose Q_B^k instead of Q_B and proceeding as before, we get

$$\det(I - sQ_B^k) = 1 - s(B_{k+1} - B_{k-1}) + s^2.$$

Since $B_{k+1} - B_{k-1} = 2C_k$ where C_k is the k^{th} Lucas-balancing number, it follows that

$$\det(I - sQ_B^k) = 1 - 2sC_k + s^2,$$

and therefore its inverse is given by

$$(I - sQ_B^k)^{-1} = \frac{1}{1 - 2sC_k + s^2} \begin{pmatrix} 1 + sB_{k-1} & -sB_k \\ sB_k & 1 - sB_{k+1} \end{pmatrix}.$$

Since $\sum_{n=0}^{\infty} s^n Q_B^{nk} = (I - sQ_B^k)^{-1}$, it follows that

$$B_{0k} + B_{1k} + sB_{2k} + s^2 B_{3k} + \ldots = \frac{B_k}{1 - 2sC_k + s^2},$$

for instance, with $k = 2$ we have

$$B_0 + B_2 + sB_4 + s^2 B_6 + \ldots = \frac{6}{1 - 34s + s^2}.$$

3 Some combinatorial identities using generating function

In this section, we will establish some new results involving balancing numbers with the help of generating function.

Theorem 3.1. *If B_n denotes the n^{th} balancing number, then*

$$B_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^i \binom{n-i-1}{i} 6^{n-2i-1}.$$

Proof. By virtue of (7), we get

$$\begin{aligned} B_0 + sB_1 + s^2 B_2 + \ldots &= s\left(1 - (6s - s^2)\right)^{-1} \\ &= s\left[1 + (6s - s^2) + (6s - s^2)^2 + \ldots\right]. \end{aligned}$$

Equating the coefficient of s^n from both the sides, we obtain

$$B_n = (6s - s^2)^{n-1}.$$

Expanding the right hand side expression of the above equation binomially we obtain the desired result. □

The proof of the following result is analogous to Theorem 3.1

Theorem 3.2. *If B_k and C_k are the k^{th} balancing and k^{th} Lucas-balancing numbers respectively then*

$$B_{nk} = B_k \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^i \binom{n-i-1}{i} (2C_k)^{n-2i-1}.$$

The following result will be shown by using Binet's formula.

Theorem 3.3. *The following identity is valid for any natural number n.*

$$B_n = \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2l+1} (\sqrt{8})^{2l} 3^{n-2l-1}.$$

Proof. Recall that $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$. Setting $\lambda_1 = x + y$ and $\lambda_2 = x - y$ so that $x = 3,\ y = \sqrt{8}$ and therefore $\frac{y}{x} = \frac{\sqrt{8}}{3}$. Using binomial theorem, we have

$$\lambda_1^n - \lambda_2^n = (x+y)^n - (x-y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k (1 - (-1)^k). \tag{8}$$

Putting $k = 2l + 1$, we observe that for $k = 0$, the right side expression vanish, so $k = 1$ if $l = 0$ and $k = n$ if $l = \frac{n-1}{2}$. Therefore (8) reduces to

$$\lambda_1^n - \lambda_2^n = \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2l+1} 2x^n \left(\frac{y}{x}\right)^{2l+1}.$$

It follows that

$$\frac{\lambda_1^n - \lambda_2^n}{2\sqrt{8}} = B_n = \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2l+1} (\sqrt{8})^{2l} 3^{n-2l-1}.$$

This completes the proof. □

We notice that, $B_0 = 0,\ B_1 = 1,\ B_2 = 6B_1 - B_0, \ldots, B_{k+2} = 6B_{k+1} - B_k$. Adding all these results, we get the following important identity.

$$B_{k+2} = 1 + 4\sum_{i=0}^{k} B_i + 5B_{k+1}.$$

4 Some congruence relations for balancing numbers

Ray has studied many congruence properties for balancing numbers and their related sequences [14, 15]. In [15], he applied some congruences identities to establish some divisibility properties of these numbers. In this section, we find some new results concerning congruences for balancing numbers.

Theorem 4.1. *For any natural number k, $B_{2k} \equiv 0\ (mod\ 2)$ and $B_{2k} \equiv 0\ (mod\ 3)$.*

Proof. Mathematical induction play the role to prove these results. Basis step is clear for the first part of the theorem as $B_0 = 0 \equiv 0\ (mod\ 2)$. Notice that $B_2 = 6 \equiv 0\ (mod\ 2)$. Assuming $B_{2m} \equiv 0\ (mod\ 2)$ for every $m \leq k$ and since $B_{k+l} = B_k B_{l+1} - B_{k-1} B_l$, we have

$$B_{2(m+1)} = B_{2m} B_3 - B_{2m-1} B_2 \equiv 0\ (mod\ 2).$$

Similarly the second part of the theorem can be proved. □

The following corollary is an immediate consequence of Theorem 4.1

Corollary 4.2. *For any natural number k, $B_{2k} \equiv 0\ (mod\ 6)$.*

Theorem 4.3. *For any natural number k, $B_{3k} \equiv 0\ (mod\ 5)$.*

Proof. The proof is analogous to Theorem 4.1. □

Therefore by virtue of Theorem 4.1 and Theorem 4.2, we have the following result.

Corollary 4.4. *For any natural number k, $B_{6k} \equiv 0\ (mod\ 10)$.*

The following result can be easily shown by induction.

Theorem 4.5. *For any natural number k, $B_{6k+2} \equiv 0\ (mod\ 2)$ and $B_{6k} \equiv 0\ (mod\ 2)$.*

Theorem 4.6. *For any odd natural number k, $B_{3k+1} \equiv 4\ (mod\ 5)$.*

Proof. By virtue of Theorem 3.3, we have

$$B_n = \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2l+1} (\sqrt{8})^{2l} 3^{n-2l-1}. \tag{9}$$

Replacing n by $3k+1$, we get

$$B_{3k+1} = \sum_{l=0}^{\lfloor \frac{3k}{2} \rfloor} \binom{3k+1}{2l+1} (\sqrt{8})^{2l} 3^{3k-2l}.$$

Notice that for every odd natural number k, $5k+4 \equiv 4\ (mod\ 5)$ and the result follows. □

Theorem 4.7. *If k is any odd natural number such that $4k+1$ is a prime, then $B_{4k+1} \equiv 4k\ (mod\ 4k+1)$.*

Proof. Let $k \in \mathbb{N}$ and k odd such that $4k+1$ is a prime. Putting $n = 4k+1$ in (9) and since $8k+1 \equiv 4k\ (mod\ 4k+1)$, we have

$$B_{4k+1} = \sum_{l=0}^{2k} \binom{4k+1}{2l+1} (\sqrt{8})^{2l} 3^{4k-2l} \equiv 8k+1 \equiv 4k\ (mod\ 4k+1).$$

This ends the proof. □

Similarly, putting $n = 4k+2$ and $n = 4k$ in (9) and since $8k+2 \equiv 0\ (mod\ 4k+1)$ and $8k-4 \equiv 4k-5\ (mod\ 4k+1)$, we have the following results.

Theorem 4.8. *If k is any odd natural number such that $4k+1$ is a prime, then $B_{4k+2} \equiv 0\ (mod\ 4k+1)$.*

Theorem 4.9. *If k is any odd natural number such that $4k+1$ is a prime, then $B_{4k} \equiv 4k-5\ (mod\ 4k+1)$.*

The following corollary is an immediate consequence of Theorem 4.8.

Corollary 4.10. *If k is any odd natural number such that $4k+1$ is a prime, then $B_{4k+2} \equiv 0\ (mod\ 4k+3)$.*

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