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An arithmetic function decreasing the natural numbers

Krassimir T. Atanassov

Department of Bioinformatics and Mathematical Modelling
IBPhBME, Bulgarian Academy of Sciences
Acad. G. Bonchev Str. Bl. 105, Sofia-1113, Bulgaria
and
Intelligent Systems Laboratory
Prof. Asen Zlatarov University, Bourgas-8000, Bulgaria
e-mail: krat@bas.bg

To Prof. Jószef Sándor for his 60th anniversary

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Abstract: A new arithmetic function is defined and some of its properties are studied.
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1 Introduction

During the last 250 years, the arithmetic functions have been an object of active research. Since the beginning of the 1980s, the author has been involved in exploring the field, introducing some arithmetic functions and studying their properties. Here, a new arithmetic function is defined and some of its basic properties are discussed.

In the beginning, some necessary definitions are given.

Let the natural number

$$n = \prod_{i=1}^{k} p_i^{\alpha_i} \tag{1}$$

be given, where $k, \alpha_1, ..., \alpha_k, k \geq 1$ are natural numbers and $p_1, ..., p_k$ are different primes.

Below, we use the following well known arithmetic functions defined for the above n as follows:

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}, \ \sigma(1) = 1,$$

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i - 1), \ \varphi(1) = 1,$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i + 1), \ \psi(1) = 1,$$

(see, e.g., [3]) and function

$$\rho(n) = \prod_{i=1}^{k}(p_i^{\alpha_i} - p_i^{\alpha_i-1} + ... + (-1)^{\alpha_i}), \ \rho(1) = 1$$

(see $[1, 2]$).

2 Main results

Let us define the arithmetic function $\downarrow$ so that for each prime number p, $\downarrow p$ is the highest prime number, smaller than p and for $n \geq 2$ given by (1):

$$\downarrow 2 = 1,$$

$$\downarrow n = \prod_{i=1}^{k}(\downarrow p_i)^{\alpha_i}.$$

For example, $\downarrow 12 = (\downarrow 2)^2 . \downarrow 3 = 1^2.2 = 2$ and $\downarrow 7 = 5$.

We see immediately that for every two different prime numbers p and q

$$\downarrow (pq) = \downarrow p \downarrow q,$$

i.e., function $\downarrow$ is a multiplicative one.

Theorem 1. For every natural number $n \geq 3$,

$$\frac{\varphi(n)}{n} > \frac{\downarrow \varphi(n)}{\downarrow n}. \tag{2}$$

Proof. Let n be a prime number. Then, obviously, for $n \geq 3$:

$$\frac{n-1}{n} > \frac{\downarrow n - 1}{\downarrow n}. \tag{3}$$

Let us assume that (2) is valid for some natural number n and let the prime number $p \notin \underline{set}(n)$. Therefore, $\downarrow p \notin \underline{set}(\downarrow n)$. Then, from (3) and by induction, we obtain:

$$\frac{\varphi(n)(p-1)}{np} - \frac{\varphi(\downarrow n)(\downarrow p - 1)}{\downarrow n \downarrow p} > 0.$$

Let the prime number $p \in \underline{set}(n)$. Then by induction

$$\frac{\varphi(n)p}{np} = \frac{\varphi(n)}{n} > \frac{\varphi(\downarrow n)}{\downarrow n} = \frac{\varphi(\downarrow n)\downarrow p}{\downarrow n \downarrow p} = \frac{\varphi(\downarrow n \downarrow p)}{\downarrow (np)} = \frac{\varphi(\downarrow (np))}{\downarrow (np)},$$

that proves Theorem 1. □

Theorem 2. For every natural number $n \geq 2$,

$$\frac{\sigma(n)}{n} < \frac{\downarrow \sigma(n)}{\downarrow n}. \tag{4}$$

Proof. Let n be a prime number. Then, obviously, for $n \geq 2$:

$$\frac{n+1}{n} < \frac{\downarrow n + 1}{\downarrow n}. \tag{5}$$

Let us assume that (4) is valid for some natural number n and let the prime number $p \notin \underline{set}(n)$. Then, from (5) and by induction, we obtain:

$$\frac{\sigma(np)}{np} = \frac{\sigma(n)(p+1)}{np} < \frac{\sigma(\downarrow n)(\downarrow p + 1)}{\downarrow n \downarrow p} = \frac{\sigma(\downarrow n \downarrow p)}{\downarrow (np)} = \frac{\sigma(\downarrow (np))}{\downarrow (np)}.$$

Let the prime number $p \in \underline{set}(n)$. Then $n = m.p^a$ for some natural numbers m and a and by induction

$$\frac{\sigma(np)}{np} = \frac{\sigma(n)}{n}.\frac{p^{a+2}-1}{p^{a+1}-1}.\frac{1}{p} < \frac{\sigma(\downarrow n)}{\downarrow n}.\frac{(\downarrow p)^{a+2}-1}{(\downarrow p)^{a+1}-1}.\frac{1}{\downarrow p},$$

because

$$\frac{(\downarrow p)^{a+2}-1}{(\downarrow p)^{a+2} - \downarrow p} - \frac{p^{a+2}-1}{p^{a+2}-p}$$

$$= \frac{1}{((\downarrow p)^{a+2} - \downarrow p)(p^{a+2}-p)}.\big(p^{a+2}(\downarrow p)^{a+2} - p^{a+2} - (\downarrow p)^{a+2}p + p - p^{a+2}(\downarrow p)^{a+2}$$
$$+p^{a+2}\downarrow p + (\downarrow p)^{a+2} - \downarrow p\big)$$

$$= \frac{1}{((\downarrow p)^{a+2} - \downarrow p)(p^{a+2}-p)}.\big(p \downarrow p(p^{a+1} - \downarrow p)^{a+1}) - p^{a+2} + (\downarrow p)^{a+2} + p - \downarrow p\big) > 0,$$

that proves Theorem 2. □

By analogy, we can prove

Theorem 3. For every natural number $n \geq 2$,

$$\frac{\psi(n)}{n} < \frac{\downarrow \psi(n)}{\downarrow n}.$$

Theorem 4. For every natural number $n \geq 3$,

$$\frac{\rho(n)}{n} > \frac{\downarrow \rho(n)}{\downarrow n}.$$

Theorem 5. For every odd number $n \geq 3$,

$$\varphi(n)\psi(\downarrow n)) > \varphi(\downarrow n)\psi(n), \tag{6}$$

$$\varphi(n)\sigma(\downarrow n)) > \varphi(\downarrow n)\sigma(n),$$

$$\rho(n)\psi(\downarrow n)) > \rho(\downarrow n)\psi(n),$$

$$\rho(n)\sigma(\downarrow n)) > \rho(\downarrow n)\sigma(n).$$

Proof. Let us prove (6). Let $n \geq 3$ be a prime number. Then,

$$(n-1)(\downarrow n+1) - (\downarrow n-1)(n+1) = 2(n-\downarrow n) > 0. \tag{7}$$

Let us assume that (6) is valid for some natural number n and let the prime number $p \notin \underline{set}(n)$. Then, from (6) and (7), and by induction, we obtain:

$$\varphi(np)\psi(\downarrow (np))) - \varphi(\downarrow (np))\psi(np)$$

$$= \varphi(n)(p-1)\psi(\downarrow n)(\downarrow p+1) - \varphi(\downarrow n)(\downarrow p-1)\psi(n)(p+1) > 0.$$

Let the prime number $p \in \underline{set}(n)$. Then, by induction

$$\varphi(np)\psi(\downarrow (np))) - \varphi(\downarrow (np))\psi(np) = \varphi(n)p\psi(\downarrow n)\downarrow p - \varphi(\downarrow n)\downarrow p\psi(n)p$$

$$= p\downarrow p(\varphi(n)\psi(\downarrow n) - \varphi(\downarrow n)p\psi(n)) > 0.$$

Theorem 5 is proven. □

3 Conclusion, or a mathematical joke

We will finish the present remark with the question: How many solutions (finite or infinite number) does the equality

$$\downarrow p = p - 2$$

have for each prime number p?

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