RETRACTION NOTICE

As of June 2021, the paper “A note on prime zeta function and Riemann zeta function” by Mladen Vassilev–Missana, published in *Notes on Number Theory and Discrete Mathematics*, Vol. 22, 2016, No. 4, 12–15, has been retracted.

The reasons for retraction are identified errors, as pointed out by Richard P. Brent in the paper “On two theorems of Vassilev-Missana”, submitted on 23 March 2021 and immediately published in the June issue of *Notes on Number Theory and Discrete Mathematics*, Vol. 27, 2021, No. 2, 49–50, DOI: 10.7546/nntdm.2021.27.2.49-50.
`http://nntdm.net/volume-27-2021/number-2/49-50/`

Correction entitled “A note on prime zeta function and Riemann zeta function. Corrigendum” was provided by Mladen Vassilev–Missana on 21 April 2021 and published in the same issue of *Notes on Number Theory and Discrete Mathematics*, Vol. 27, 2021, No. 2, 51–53, DOI: 10.7546/nntdm.2021.27.2.51-53.
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The electronic copy of the retracted paper is retained on the Journal website in order to maintain the scientific record, with the additional “Retracted Paper” watermark and the accompanying Retraction Notice, which as of June 2021 must be considered an integral part of the publication.

The Publisher apologises for any inconvenience caused!

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A note on prime zeta function and Riemann zeta function

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Abstract. In the present paper, we first deduce a new recurrent formula, that connects $P(s)$, $P(2s)$ and $\zeta(s)$, where $P(s)$ is the prime zeta function and $\zeta(s)$ is Riemann zeta function. After that, with the help of this recurrent formula, we find a new formula for $P(s)$ expressing $P(s)$ as infinite nested radicals (roots), depending on the values of $\zeta(2^k s)$ for $k = 0, 1, 2, 3, \ldots$.
Keywords: Prime zeta function, Riemman zeta function, prime numbers
AMS Classification: 11A25, 11M06

Dedicated to Prof. J. Sándor
on the occasion of his 60th birthday

1 Introduction

As usual, we have

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \tag{1}$$

where $s \in \mathcal{C}$ ($\mathcal{C}$ is the set of complex numbers) and $\Re(s) > 1$. So, $\zeta(s)$ is the so-called Riemann zeta function.

Also, by $P(s)$ we mean:

$$P(s) = \sum_{p} \frac{1}{p^s}, \tag{2}$$

where the sum is taken over all primes. So, $P(s)$ is the so-called prime zeta function. For $P(s)$ there is a well known representation

$$P(s) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \ln(\zeta(ks)),$$

(see [1, 2, 3]).

Below, we find a new representation for $P(s)$.

Let $\mathcal{P}$ be the set of all primes and $\overline{\mathcal{P}}$ be the set of all composite numbers $m > 1$. We set

$$\overline{P}(s) = \sum_{m} \frac{1}{m^s}, \tag{3}$$

where m is taken over $\overline{\mathcal{P}}$.

From (1), (2) and (3) we have obviously:

$$P(s) + \overline{P}(s) = \zeta(s) - 1. \tag{4}$$

Our aim is to express $\overline{P}(s)$ with the help of $P(s)$ and $\zeta(s)$. By this reason, we consider the product

$$P(s).\sum_{i=2}^{\infty} \frac{1}{i^s}, \tag{5}$$

i.e., the product

$$I_1.I_2 = \left(\sum_{p} \frac{1}{p^s}\right).\left(\sum_{l=2}^{\infty} \frac{1}{i^s}\right). \tag{6}$$

It is clear that the multiplication of I_1 and I_2 yields $\overline{\mathcal{P}}$, but also, some repeating terms will be there.

Below, we describe these repeating terms under the condition that the integer $s > 1$.

Let $p \in I_1$ and $a \in I_2$ be such ones that for $q \in I_1, q \neq p$ and $b \in I_2$ we have

$$\frac{1}{p^s}.\frac{1}{a^s} = \frac{1}{q^s}.\frac{1}{b^s}.$$

Then

$$p^s.a^s = q^s.b^s.$$

Hence, p divides b and therefore, $b = kp$. Replacing the last into the above equality, we obtain:

$$a^s = k^s.q^s.$$

Hence, q divides a and therefore $a = tq$. Hence

$$t^s = k^s.$$

Therefore, $t = k$. Hence, $a = kq$, $b = kp$, where $p \neq q$ are primes and $k = 1, 2, 3, ...$

Now, it is clear that the expression

$$S = \left(\sum_{k=1}^{\infty} \frac{1}{k^s}\right)\left(\sum \frac{1}{p^s q^s}\right), \tag{7}$$

where $\sum \frac{1}{p^s q^s}$ is the sum taken over all primes p and q, for which $p \neq q$, is meeting twice. Hence,

$$\overline{P}(s) = I_1.I_2 - S. \tag{8}$$

We may rewrite (7) in the form

$$S = \zeta(s).J_2^{(s)}, \tag{9}$$

where

$$J_2^{(s)} = \sum \frac{1}{p^s q^s}. \tag{10}$$

Now, from (5), (6), (9) and (10) we obtain

$$\overline{P}(s) = P(s).(\zeta(s) - 1) - \zeta(s).J_2^{(s)}. \tag{11}$$

Replacing (11) into (4), we obtain

$$\zeta(s).P(s) - \zeta(s).J_2^{(s)} = \zeta(s) - 1. \tag{12}$$

Solving (12) with respect to $P(s)$, we obtain

$$P(s) = 1 - \frac{1}{\zeta(s)} + J_2^{(s)}. \tag{13}$$

It remains only to find $J_2^{(s)}$, but we have:

$$(P(s))^2 = \left(\sum_p \frac{1}{p^s}\right)^2 = \sum_p \frac{1}{p^{2s}} + 2J_2^{(s)} \tag{14}$$

and

$$\sum_p \frac{1}{p^{2s}} = P(2s). \tag{15}$$

From (14) and (15) it follows:

$$J_2^{(s)} = \frac{(P(s))^2 - P(2s)}{2}. \tag{16}$$

Using (13) and (16), we obtain easily

$$(1 - P(s))^2 = \frac{2}{\zeta(s)} - 1 + P(2s).$$

So, we prove the following important result.

Theorem 1. *For integer $s > 1$, the following recurrent relation*

$$(1 - P(s))^2 = \frac{2}{\zeta(s)} - 1 + P(2s). \tag{17}$$

holds.

Corollary 1. *For integer $s > 1$, the following recurrent relation*

$$P(s) = 1 - \sqrt{\frac{2}{\zeta(s)} - 1 + P(2s)}\,. \tag{18}$$

holds.

The proof is obvious from (17) and the fact that $0 < P(s) < 1$.

Now, we make a remarkable application of (18). These equalities yield:

$$P(2s) = 1 - \sqrt{\frac{2}{\zeta(2s)} - 1 + P(4s)}\,,$$

$$P(4s) = 1 - \sqrt{\frac{2}{\zeta(4s)} - 1 + P(8s)}\,,$$

$$P(8s) = 1 - \sqrt{\frac{2}{\zeta(8s)} - 1 + P(16s)}\,,$$

etc.

Using these infinitely many equalities and putting each of them into the previous one, we come to our main result.

Theorem 2. *Prime zeta function $P(s)$, for every fixed integer $s > 1$, could be expressed with the help of the values of Riemann zeta function: $\zeta(2^k.s), k = 0, 1, 2, 3, \ldots$, by the formula:*

$$P(s) = 1 - \sqrt{\frac{2}{\zeta(s)} - \sqrt{\frac{2}{\zeta(2s)} - \sqrt{\frac{2}{\zeta(4s)} - \sqrt{\frac{2}{\zeta(8s)} \cdots}}}}\,.$$

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