Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 22, 2016, No. 3, 45–53

On sum and ratio formulas for balancing-like sequences

Ravi Kumar Davala and G. K. Panda

Department of Mathematics, National Institute of Technology
Rourkela, India
e-mails: davalaravikumar@gmail.com,
gkpanda_nit@rediffmail.com

Received: 15 August 2015 **Revised:** 24 August 2016 **Accepted:** 27 September 2016

Abstract: Certain sum formulas with terms from balancing-like and Lucas-balancing-like sequences are discussed. The resemblance of some of these formulas with corresponding sum formulas involving natural numbers are exhibited.

Keywords: Balancing-like sequences, Lucas-balancing-like sequences, Fibonacci sequence, Binet form.
AMS Classification: 11B39.

1 Introduction

Balancing numbers are terms of the binary recurrence $x_{n+1} = 6x_n - x_{n-1}$ with initial values $x_0 = 0, x_1 = 1$ (see [6]). Subsequently, Panda and Rout [5] studied sequences obtained from the class of binary recurrences $G_{n+1} = AG_n - BG_{n-1}$ with initial values $G_0 = 0, G_1 = 1$ where A and B are natural numbers satisfying $A^2 - 4B > 0$. They showed that if $B = 1$, the sequence generated from the above class of binary recurrences behave like the balancing sequence and hence they called such sequences as balancing-like sequences.

Recently, Davala and Panda [1] established several sum and ratio formulas involving balancing numbers. The objective of this work is to establish such results with terms from balancing-like sequence and their associate sequences.

Let $a \geq 3$ be a fixed positive integer. The associate sequence corresponding to the balancing-like sequence $G_{n+2} = aG_{n+1} - G_n, G_0 = 0, G_1 = 1$ is the binary recurrence $H_{n+2} = aH_{n+1} -$

$H_n,\ H_0 = 2,\ H_1 = a$, and is called Lucas-balancing-like sequence [7, p.26]. It is easy to see that the Binet forms for the sequences $\{G_n\}$ and $\{H_n\}$ are respectively

$$G_n = \frac{\alpha^n - \beta^n}{\sqrt{a^2-4}},\ \ H_n = \alpha^n + \beta^n,$$

where, $\alpha = \frac{a+\sqrt{a^2-4}}{2}$ and $\beta = \frac{a-\sqrt{a^2-4}}{2}$.

2 Identities involving balancing-like and Lucas balancing-like sequences

In this section, we establish certain sum formulas in which the summands involve terms from the balancing-like sequences. The following identities involving terms from the the balancing-like and their associate sequences are needed for the subsequent development of this work.

(a) $G_{-n} = -G_n,\ H_{-n} = H_n$

(b) $G_{2n} = G_n H_n$.

(c) ${H_n}^2 - H_{2n} = 2$

(d) $H_{2n} - (a^2-4){G_n}^2 = 2$

(e) $2G_{n\pm m} = G_n H_m \pm G_m H_n$

(f) $2H_{n\pm m} = H_n H_m \pm (a^2-4) G_n G_m$.

We do not provide proofs of these identities as they can be easily proved using Binet forms for the sequences $\{G_n\}$ and $\{H_n\}$. These identities will be used subsequently without further reference.

In a recent paper [1], the authors have established the formula $B_{n+2} = 5B_{n+1} + 4\sum_{k=1}^{n} B_i + 1$ relating to the balancing sequence. The following identities generalize the above formula to balancing-like and Lucas-balancing-like sequences. Their proofs are similar to that of Theorem 1.1 of [1].

$G_{n+2} = (a-1)G_{n+1} + (a-2)\sum_{k=1}^{n} G_i + 1,$

$H_{n+2} = (a-1)H_{n+1} + (a-2)\sum_{k=1}^{n} H_i + a - 2.$

Using the above identities and mathematical induction, it is easy to prove the following theorem dealing with weighted partial sums of balancing-like and Lucas-balancing-like sequences.

Theorem 2.1. *For* $n \in \mathbb{N}$

(a) $nG_1 + (n-1)G_2 + \cdots + 2G_{n-1} + G_n = \frac{1}{a-2}\left[G_{n+1} - (n+1)\right],$

(b) $nH_1 + (n-1)H_2 + \cdots + 2H_{n-1} + H_n = \frac{1}{a-2}\left[H_{n+1} - (n+1)(a-2) + 2\right].$

In the next theorem, we consider finding partial weighted sums involving terms from the sequences $\{G_n\}$ and $\{H_n\}$ with binomial coefficients as weights.

Theorem 2.2. *For* $n \in \mathbb{N}$

(a) $\sum_{i=0}^{n} \binom{n}{i} G_i H_{n-i} = 2^n G_n,$

(b) $\sum_{i=0}^{n} \binom{n}{i} G_{n+1-2i} = a^n,$

(c) $\sum_{i=0}^{n} (-1)^i \binom{n}{i} G_i H_{n-i} = \begin{cases} 0, & if\ n\ is\ even, \\ -2(a^2-4)^{\frac{n-1}{2}}, & if\ n\ is\ odd. \end{cases}$

(d) $\sum_{i=0}^{n} (-1)^i \binom{n}{i} H_{n+1-2i} = \begin{cases} a(a^2-4)^{\frac{n}{2}}, & if\ n\ is\ even, \\ (a^2-4)^{\frac{n+1}{2}}, & if\ n\ is\ odd. \end{cases}$

Proof. We will prove (a) and (c) only. The proof of (b) and (d) are similar.

Since $\alpha - \beta = \sqrt{a^2-4}$, we have

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} G_i H_{n-i} =& \frac{1}{\sqrt{a^2-4}} \binom{n}{i} [\alpha^n - \beta^n + \beta^n \alpha^{2i} - \alpha^n \beta^{2i}] \\
=& G_n \sum_{i=0}^{n} \binom{n}{i} + \frac{1}{\sqrt{a^2-4}} \left[\beta^n \sum_{i=0}^{n} \alpha^{2i} - \alpha^n \sum_{i=0}^{n} \beta^{2i} \right] \\
=& 2^n G_n + \frac{1}{\sqrt{a^2-4}} [\beta^n (1+\alpha^2)^n - \alpha^n (1+\beta^2)^n] \\
=& 2^n G_n + \frac{1}{\sqrt{a^2-4}} [(\beta+\alpha)^n - (\alpha+\beta)^n] \\
=& 2^n G_n,
\end{aligned}$$

which proves (a). Further,

$$\begin{aligned}
\sum_{i=0}^{n} (-1)^i \binom{n}{i} G_i H_{n-i} =& \frac{1}{\sqrt{a^2-4}} (-1)^i \binom{n}{i} [\alpha^n - \beta^n + \beta^n \alpha^{2i} - \alpha^n \beta^{2i}] \\
=& \frac{1}{\sqrt{a^2-4}} \left[\beta^n \sum_{i=0}^{n} (-1)^i \alpha^{2i} - \alpha^n \sum_{i=0}^{n} (-1)^i \beta^{2i} \right] \\
=& \frac{1}{\sqrt{a^2-4}} [\beta^n (1-\alpha^2)^n - \alpha^n (1-\beta^2)^n] \\
=& \frac{1}{\sqrt{a^2-4}} [(\beta-\alpha)^n - (\alpha-\beta)^n] \\
=& -\frac{1}{\sqrt{a^2-4}} (\alpha-\beta)^n [1 - (-1)^n] \\
=& \begin{cases} 0 & \text{if } n \text{ is even}, \\ -2(a^2-4)^{\frac{n-1}{2}} & \text{if } n \text{ is odd}, \end{cases}
\end{aligned}$$

and (c) is proved. $\square$

The following corollary is an immediate consequence of Theorem 2.2 (c).

Corollary 2.1. *For* $n \in \mathbb{N}$

(a) $\binom{2n}{0}G_{2n} - \binom{2n}{1}G_{2n-2} + \cdots + (-1)^n\binom{2n}{n}G_0 = 0,$

(b) $\binom{2n+1}{0}G_{2n+1} - \binom{2n+1}{1}G_{2n-1} + \cdots + (-1)^n\binom{2n+1}{n}G_1 = (a^2-4)^n.$

3 Variants of Melham identities for balancing-like sequences and Lucas-balancing-like sequences

Using the basic identities for Fibonacci and Lucas numbers, Melham [3, 4] managed to find several sum and alternating sum formulas of certain products of Fibonacci and Lucas numbers. Motivated by these results, we obtain certain variants of these results for balancing-like sequences and establish their resemblance with formulas involving natural numbers.

The following theorem, which contains index reduction formulas, will play crucial role for establishing results of this section.

Theorem 3.1. *If* x, y, z, w *and* r *are integers and* $x + y = z + w$ *then*

(a) $G_{x+r}H_{y+r} - G_{z+r}H_{w+r} = G_xH_y - G_zH_w,$

(b) $G_{x+r}G_{y+r} - G_{z+r}G_{w+r} = G_xG_y - G_zG_w,$

(c) $H_xH_y - H_zH_w = -(a^2-4)[G_xG_y - G_zG_w].$

Proof. We provide the proof of (a) only. The proofs of (b) and (c) are similar.

Using the Binet forms for the sequences $\{G_n\}$ and $\{H_n\}$, we get

$$\begin{aligned}
&G_{x+r}H_{y+r} - G_{z+r}H_{w+r}\\
&= \frac{1}{\sqrt{a^2-4}}[(\alpha^{x+r} - \beta^{x+r})(\alpha^{y+r} + \beta^{y+r}) - (\alpha^{z+r} - \beta^{z+r})(\alpha^{w+r} + \beta^{w+r})]\\
&= \frac{1}{\sqrt{a^2-4}}(\alpha^{x+y+2r} - \beta^{x+y+2r} + \alpha^{x+r}\beta^{y+r} - \beta^{x+r}\alpha^{y+r})\\
&\quad - \frac{1}{\sqrt{a^2-4}}(\alpha^{z+w+2r} - \beta^{z+w+2r} + \alpha^{z+r}\beta^{w+r} - \beta^{z+r}\alpha^{w+r})\\
&= \frac{1}{\sqrt{a^2-4}}[(\alpha^x\beta^y - \beta^x\alpha^y) - (\alpha^z\beta^w - \beta^z\alpha^w)]\\
&= \frac{1}{\sqrt{a^2-4}}[(\alpha^{x+y} - \beta^{x+y+} + \alpha^x\beta^y - \beta^x\alpha^y) - (\alpha^{z+w} - \beta^{z+w} + \alpha^z\beta^w - \beta^z\alpha^w)]\\
&= \frac{1}{\sqrt{a^2-4}}[(\alpha^x - \beta^x)(\alpha^y + \beta^y) - (\alpha^z - \beta^z)(\alpha^w + \beta^w)]\\
&= G_xH_y - G_zH_w.
\end{aligned}$$

□

The following two theorems deal with sum formulas for certain products with terms from a balancing-like sequence and its corresponding Lucas-balancing-like sequence.

Theorem 3.2. *Let* s, n *and* m *are positive integers, then*

$$\sum_{k=1}^{n} G_{sk}G_{s(k+1)}\cdots G_{s(k+2m)}H_{s(k+m)} = \frac{G_{sn}G_{s(n+1)}\cdots G_{s(n+2m+1)}}{G_{s(m+1)}}. \tag{3.1}$$

Proof. Let l_n and r_n denote the left and right sides respectively, of (3.1). Then

$$\begin{aligned}
r_n - r_{n-1} &= \frac{G_{sn}G_{s(n+1)}\cdots G_{s(n+2m)}}{G_{s(m+1)}}\,[G_{s(n+2m+1)} - G_{s(n-1)}]\\
&= \frac{G_{sn}G_{s(n+1)}\cdots G_{s(n+2m)}}{G_{s(m+1)}}\,[G_{s(n+m)+s(m+1)} - G_{s(n+m)-s(m+1)}]\\
&= \frac{G_{sn}G_{s(n+1)}\cdots G_{s(n+2m)}}{G_{s(m+1)}}\,[H_{s(n+m)}G_{s(m+1)}]\\
&= l_n - l_{n-1}.
\end{aligned}$$

Now,

$$r_1 = \frac{G_sG_{2s}\cdots G_{s(2m+1)}G_{s(2m+2)}}{G_{s(m+1)}} = G_sG_{2s}\cdots G_{s(2m+1)}H_{s(m+1)} = l_1. \qquad \square$$

For any natural number n, $\sum_{k=1}^{n} 2k = n(n+1)$ and $\sum_{k=1}^{n} k(k+2)(2k+2) = \frac{n(n+1)(n+2)(n+3)}{2}$ are well known. The identities in the following corollary are consequence of Theorem 3.2 and resembles the identities just stated.

Corollary 3.1. *If* $n \in \mathbb{N}$

(a) $\sum_{k=1}^{n} G_{2k} = G_nG_{n+1}$,

(b) $\sum_{k=1}^{n} G_kG_{k+2}G_{2k+2} = \frac{G_nG_{n+1}G_{n+2}G_{n+3}}{G_2}$.

Theorem 3.3. *If* s, n *and* m *are positive integers, then*

$$\sum_{k=1}^{n} H_{sk}H_{s(k+1)}\cdots H_{s(k+2m)}G_{s(k+m)} = \frac{H_{sn}H_{s(n+1)}\cdots H_{s(n+2m+1)}}{(a^2-4)G_{s(m+1)}} - R_0,$$

where

$$R_n = \frac{H_{sn}H_{s(n+1)}\cdots H_{s(n+2m+1)}}{(a^2-4)G_{s(m+1)}}.$$

Proof. Since

$$\begin{aligned}
R_n - R_{n-1} &= \frac{H_{sn}H_{s(n+1)}\cdots H_{s(n+2m)}}{(a^2-4)G_{s(m+1)}}\,[H_{s(n+2m+1)} - H_{s(n-1)}]\\
&= \frac{H_{sn}H_{s(n+1)}\cdots H_{s(n+2m)}}{(a^2-4)G_{s(m+1)}}\,[H_{s(n+m)+s(m+1)} - H_{s(n+m)-s(m+1)}]\\
&= \frac{H_{sn}H_{s(n+1)}\cdots H_{s(n+2m)}}{(a^2-4)G_{s(m+1)}}\,[(a^2-4)G_{s(n+m)}G_{s(m+1)}]\\
&= l_n - l_{n-1},
\end{aligned}$$

it follows that $l_n - R_n = c$, where c is a constant. Now,

$$\begin{aligned}
c &= l_1 - R_1\\
&= H_sH_{2s}\cdots H_{s(1+2m)}G_{s(1+m)} - \frac{H_sH_{2s}\cdots H_{s(2+2m)}}{(a^2-4)G_{s(m+1)}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{H_s H_{2s} \cdots H_{s(1+2m)}}{(a^2-4)G_{s(m+1)}} \left[(a^2-4)G_{s(m+1)}^2 - H_{s(2m+2)}\right] \\
&= \frac{-H_0 H_s H_{2s} \cdots H_{s(1+2m)}}{(a^2-4)G_{s(m+1)}} = -R_0.
\end{aligned}
$$

This ends the proof. $\square$

In the remaining part of this work, we use the notation $[F(k)]_a^b = F(b) - F(a)$. In the following theorem, we state alternative sum formulas for certain products with terms from Lucas-balancing-like sequence and its corresponding balancing-like sequence. The proof of the following theorem is omitted as its proof is similar to that of the Theorems 3.2 and 3.3.

Theorem 3.4. *If s, n and m are positive integers then*

1. $\displaystyle\sum_{k=1}^{n}(-1)^k G_{sk}G_{s(k+1)}\cdots G_{s(k+2m)}G_{s(k+m)} = (-1)^n \frac{G_{sn}G_{s(n+1)}\cdots G_{s(n+2m+1)}}{H_{s(m+1)}}$,
2. $\displaystyle\sum_{k=1}^{n}(-1)^k H_{sk}H_{s(k+1)}\cdots H_{s(k+2m)}H_{s(k+m)} = \left[\frac{(-1)^k H_{sk}H_{s(k+1)}\cdots H_{s(k+2m+1)}}{H_{s(m+1)}}\right]_{k=0}^{k=n}$,
3. $\displaystyle\sum_{k=1}^{n}G_{sk}^2 G_{s(k+1)}^2\cdots G_{s(k+2m)}^2 G_{s(2k+2m)} = \frac{G_{sn}^2 G_{s(n+1)}^2\cdots G_{s(n+2m+1)}^2}{G_{s(2m+2)}}$,
4. $\displaystyle\sum_{k=1}^{n}H_{sk}^2 H_{s(k+1)}^2\cdots H_{s(k+2m)}^2 G_{s(2k+2m)} = \left[\frac{H_{sk}^2 H_{s(k+1)}^2\cdots H_{s(k+2m+1)}^2}{(a^2-4)G_{s(2m+2)}}\right]_{k=0}^{k=n}$.

One can easily verify the sum formulas $\sum_{k=1}^{n}(-1)^{n-k}k^2 = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n}k^2\cdot 2k = \frac{n^2(n+1)^2}{2}$. The identities in the following corollary, which follows from (1) and (3) of Theorem 3.4, bears resemblance to the above identities.

Corollary 3.2. *For $n \in \mathbb{N}$*

(a) $\sum_{k=1}^{n}(-1)^{n-k}G_k^2 = \frac{G_n G_{n+1}}{G_2}$,

(b) $\sum_{k=1}^{n}G_k^2 G_{2k} = \frac{G_n^2 G_{n+1}^2}{G_2}$.

The identity $\sum_{k=1}^{n}(-1)^{n-k}(2k-1) = n$ can be written as $\sum_{k=1}^{n}(-1)^{n-k}(2k-1)2 = 2n$. In the following theorem, we state an identity (which can be easily proved by mathematical induction) involving terms from a balancing-like sequence and resembling the last identity.

Theorem 3.5. *For $n \in \mathbb{N}$,* $\sum_{k=1}^{n}G_{[(-1)^{n-k}(2k-1)]}G_2 = \sum_{k=1}^{n}(-1)^{n-k}G_{2k-1}G_2 = G_{2n}$.

The sum formula $\sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{6}$ is well-known and can be written as $\sum_{k=1}^{n}k\cdot 3k = \frac{n(n+1)(2n+1)}{2}$ and in the following theorem, we prove an identity with terms involving balancing-like sequence resembling the last one.

Theorem 3.6. *For $n \in \mathbb{N}$,* $\sum_{k=1}^{n}G_k G_{3k} = \frac{G_n G_{n+1} G_{2n+1}}{G_2}$.

Proof. We will use mathematical induction on n. The identity is obviously true for $n = 1$. Assume that the identity is true for $n = m$. For $n = m + 1$,

$$\begin{aligned}\sum_{k=1}^{m+1} G_k G_{3k} &= \sum_{k=1}^{n} G_k G_{3k} + G_{m+1} G_{3m+3} \\ &= \frac{G_m G_{m+1} G_{2m+1}}{G_2} + G_{m+1} G_{3m+3} \\ &= \frac{G_{m+1}}{G_2} \left[G_m G_{2m+1} + G_2 G_{3m+3} \right] = \frac{G_{m+1}}{G_2} \left[G_{m+2} G_{2m+3} \right].\end{aligned}$$

This ends the proof. □

It is easy to see that $\sum_{k=0}^{n-1} \frac{1}{2^{k+1}} = \frac{2^n - 1}{2^n}$. In the following theorem, we prove an analogous identity in terms of balancing-like sequences.

Theorem 3.7. $\displaystyle\sum_{k=0}^{n-1} \frac{1}{G_{2^{k+1}}} = \frac{G_{2^n - 1}}{G_{2^n}}$.

Proof.

$$\begin{aligned}\sum_{k=0}^{n-1} \frac{1}{G_{2^{k+1}}} &= \sum_{k=0}^{n-1} \frac{G_{2^{k+1} - 2^k}}{G_{2^{k+1}} G_{2^k}} = \sum_{k=0}^{n-1} \frac{G_{2^{k+1}} G_{2^k + 1} - G_{2^{k+1} + 1} G_{2^k}}{G_{2^{k+1}} G_{2^k}} \\ &= \sum_{k=0}^{n-1} \left[\frac{G_{2^k + 1}}{G_{2^k}} - \frac{G_{2^{k+1} + 1}}{G_{2^k}} \right] = \frac{G_2}{G_1} - \frac{G_{2^n + 1}}{G_{2^n}} \\ &= \frac{G_2 G_{2^n} - G_1 G_{2^n + 1}}{G_{2^n}} = \frac{G_{2^n - 1}}{G_{2^n}}.\end{aligned}$$

□

Good [2] proved the Fibonacci identity

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7 - \sqrt{5}}{2}.$$

It is easy to see that $\lim_{n \to \infty} \frac{G_{n+1}}{G_n} = \alpha$, where $\alpha = \frac{a + \sqrt{a^2 - 4}}{2}$. The following corollary, which follows from the previous theorem and resembles the above identity of Good, links an infinite sum of reciprocals of a subsequence of a balancing-like sequence to α.

Corollary 3.3. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{G_{2^{k+1}}} = \frac{1}{\alpha}$.

In [1], the authors established the formula

$$\frac{B_{m+2n+2k} - B_m}{B_{m+n+2k} - B_{m+n}} = \frac{B_{n+k}}{B_k}$$

involving balancing numbers. The following identities generalize the above formula to balancing-like and Lucas-balancing-like sequences.

Theorem 3.8. *For* m, n *and* $k \in \mathbb{N}$, $\displaystyle\frac{H_{m+2n+2k} - H_m}{H_{m+n+2k} - H_{m+n}} = \frac{G_{n+k}}{G_k} = \frac{G_{m+2n+2k} - G_m}{G_{m+n+2k} - G_{m+n}}$.

Proof. From Theorem 3.1 (a), we have

$$G_{n+k}H_{n+m} - G_kH_m = G_nH_{n+m+k} = G_{n+k}H_{n+m+2k} - G_kH_{n+2m+2k},$$

from which we get

$$\frac{H_{m+2n+2k} - H_m}{H_{m+n+2k} - H_{m+n}} = \frac{G_{n+k}}{G_k}.$$

Further, from Theorem 3.1 (b), we have

$$G_{k+n}G_{m+n} - G_mG_k = G_nG_{n+m+k} = G_{n+k}G_{n+m+2k} - G_kG_{n+2m+2k}$$

and rearrangement gives

$$\frac{G_{n+k}}{G_k} = \frac{G_{m+2n+2k} - G_m}{G_{m+n+2k} - G_{m+n}}.$$

□

In the next two theorems, we present certain quotients involving sums or differences of balancing-like numbers and Lucas-balancing-like numbers simplifying to linear expressions of balancing-like numbers and Lucas-balancing-like numbers. In some cases, the subscripts of balancing-like numbers and Lucas-balancing-like numbers involved in the quotients are in arithmetic progressions.

Theorem 3.9. *For $m, n \in \mathbb{N}$, each of $\frac{H_{m+2n+1} \pm H_m}{H_{m+n+1} \pm H_{m+n}}$ and $\frac{H_{m+3n} \pm H_m}{H_{m+2n} \pm H_{m+n}}$ are independent of m. Also the following identities hold.*

(a) $\dfrac{H_{m+2n+1} - H_m}{H_{m+n+1} - H_{m+n}} = G_{n+1} + G_n,$

(b) $\dfrac{H_{m+2n+1} + H_m}{H_{m+n+1} + H_{m+n}} = G_{n+1} - G_n,$

(c) $\dfrac{H_{m+3n} - H_m}{H_{m+2n} - H_{m+n}} = H_n + 1,$

(d) $\dfrac{H_{m+3n} + H_m}{H_{m+2n} + H_{m+n}} = H_n - 1.$

Proof. We will prove (a) and (d) only. The proof of (b) and (c) are similar.

Observe that

$$\begin{aligned}(H_{m+n+1} - H_{m+n})(G_{n+1} + G_n) =& (H_{m+n+1}G_{n+1} - H_{m+n}G_n) + (G_nH_{m+n+1} - G_{n+1}H_{m+n})\\ =& H_{m+2n+1} - H_m\end{aligned}$$

from which (a) follows. Further (d) follows from

$$\begin{aligned}(2H_{m+2n} + 2H_{m+n})(H_n - 1) =& 2H_{m+2n}H_n + 2H_{m+n}H_x - 2H_{m+2n} - 2H_{m+n}\\ =& 2H_{m+2n}H_n + H_{m+n}H_n - (a^2 - 4)G_nG_{m+n} - 2H_{m+n}\\ =& 2H_{m+2n}H_x + 2H_m - 2H_{m+n}\\ =& 2H_{m+3n} + 2H_m.\end{aligned}$$

□

The proof of the following theorem is omitted since it is similar to that of the previous theorem.

Theorem 3.10. *For* $m, n \in \mathbb{N}$, *each of* $\dfrac{G_{m+2n+1} \pm G_m}{G_{m+n+1} \pm G_{m+n}}$ *and* $\dfrac{G_{m+3x} \pm G_m}{G_{m+2n} \pm G_{m+n}}$ *are independent of* m. *Also the following identities hold.*

(a) $\dfrac{G_{m+2n+1} - G_m}{G_{m+n+1} - G_{m+n}} = G_{n+1} + G_n,$

(b) $\dfrac{G_{m+2n+1} + G_m}{G_{m+n+1} + G_{m+n}} = G_{n+1} - G_n,$

(c) $\dfrac{G_{m+3n} - G_m}{G_{m+2n} - G_{m+n}} = H_n + 1,$

(d) $\dfrac{G_{m+3n} + G_m}{G_{m+2n} + G_{m+n}} = H_n - 1.$

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