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On subgroups of non-commutative general rhotrix group

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Abstract: This paper considers the pair $(GR_n(F),\circ)$ consisting of the set of all invertible rhotrices of size *n* over an arbitrary field *F*; and together with the binary operation of row-column method for rhotrix multiplication; in order to introduce it as the concept of "*non-commutative general rhotrix group*". We identify a number of subgroups of $(GR_n(F),\circ)$ and then advance to show that its particular subgroup is embedded in a particular subgroup of the well-known general linear group $(GL_n(F),\cdot)$. Furthermore, we shall investigate isomorphic relationship between some subgroups of $(GR_n(F),\circ)$.

Keywords: Rhotrix, Matrix, Group, Subgroup, General rhotrix group, General linear group.
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1 Introduction

Rhotrix theory deals with the study of algebra and analysis of array of numbers in rhomboid shape. Since the introduction of the theory by Ajibade [1] as an extension of ideas on matrix-tertions and matrix-noitrets suggested by Atanassov and Shannon [2], there have been many demonstration of interest by researchers in the usage of rhotrix set as an underlying set in the study of various forms of algebraic structures (see [3–6]). The addition and multiplication for heart-based rhotrices of size 3 were defined in [1]. Sani [4] defined a rhotrix *R* of size *n* as a rhomboidal array of numbers which can be expressed as a couple of two square matrices *A* and *C* of sizes $(t\times t)$ and $(t-1)\times(t-1)$, where $t=\dfrac{n+1}{2}$ and $n\in 2Z^{+}+1$. That is,

$$
R_n = \left\langle A_{t\times t}, C_{(t-1)\times(t-1)} \right\rangle = \left\langle \begin{array}{ccccccc}
 & & & a_{11} & & & \\
 & & a_{21} & c_{11} & a_{12} & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \\
a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\
 & & & a_{tt} & & &
\end{array} \right\rangle
$$

$$
= \left\langle \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1(t-1)} & a_{1t} \\
a_{21} & a_{22} & \cdots & a_{2(t-1)} & a_{2t} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
a_{(t-1)1} & a_{(t-1)2} & \cdots & a_{(t-1)(t-1)} & a_{(t-1)t} \\
a_{t1} & a_{t2} & \cdots & a_{t(t-1)} & a_{tt}
\end{bmatrix}, \begin{bmatrix}
c_{11} & \cdots & c_{1(t-1)} \\
\cdots & \cdots & \cdots \\
c_{(t-1)1} & \cdots & c_{(t-1)(t-1)}
\end{bmatrix} \right\rangle,
$$

where $[a_{ij}]$ and $[c_{lk}]$ are called the major and minor matrices of R_n respectively. The set of all such collections of rhotrices with entries from an arbitrary field F is given as

$$
R_n(F) = \left\{ \left\langle \begin{array}{ccccccc}
 & & & a_{11} & & & \\
 & & a_{21} & c_{11} & a_{12} & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \\
a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\
 & & & a_{tt} & & &
\end{array} \right\rangle : a_{ij} \in F, c_{lk} \in F \right\},
$$

where $1 \le i, j \le t$, $1 \le l, k \le t-1$; $t = \dfrac{n+1}{2}$ and $n \in 2Z^{+} + 1$.

A row-column method for multiplication of two rhotrices R_n, Q_n having the same size was defined by Sani [4] as follows:

$$
R_n \circ Q_n = \left\langle a_{i_1 j_1}, c_{l_1 k_1} \right\rangle \circ \left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle = \left\langle \sum_{i_2 j_1}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle .
$$

It was noted in [4] that this rhotrix multiplication is non-commutative but associative. The identity rhotrix for any real rhotrix of size n was given as

$$
I_n = \left\langle I_{t\times t}, I_{(t-1)\times(t-1)} \right\rangle = \left\langle \begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & 1 & 0 & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \\
0 & \cdots & \cdots & 1 & \cdots & \cdots & 0 \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & 0 & 1 & 0 & & \\
 & & & 1 & & &
\end{array} \right\rangle .
$$

It was also stated in [4] that since R_n can be represented as $R_n = \langle a_{ij}, c_{lk}\rangle$; if both matrices $[a_{ij}]$ and $[c_{lk}]$ are invertible, then R_n is invertible and $R_n^{-1} = \langle q_{ij}, r_{lk}\rangle$, where q_{ij} and r_{lk} are the inverse entries of $A_{t\times t}$ and $C_{(t-1)\times(t-1)}$, respectively.

The determinant of a rhotrix R of size n was also defined as $\det(R_n) = \det\langle a_{ij}, c_{lk}\rangle = \det(A_{t\times t})\cdot\det(C_{(t-1)\times(t-1)})$; and that R_n is invertible if and only if $\det(R_n) \neq 0$. Furthermore, for any rhotrix $R_n = \langle a_{ij}, c_{lk}\rangle$, the transpose of R_n was defined in [4] as $R_n^T = \langle a_{ji}, c_{kl}\rangle$. It was also shown in [4] that

$$\det(R_n \circ Q_n) = \det(R_n)\circ\det(Q_n) = \det(R_n)\cdot\det(Q_n)$$

and

$$(R_n \circ Q_n)^T = (Q_n)^T \circ (R_n)^T.$$

It was noted in [4] that the set of all invertible rhotrices of size n with entries in set of real numbers together with the binary operation of row and column method of rhotrix multiplication is a group. This idea of a rhotrix group given in [4] provides us with the motivation to consider its generalization for our study under the class of non-commutative general rhotrix group of size n over an arbitrary field F. The name results from the non-commutative but associative property of the row-column multiplication method.

In this paper, we shall adopt the row-column method for rhotrix multiplication in order to consider an algebraic study of non-commutative groups of rhotrices and their generalization. This will be achieved through our consideration of the pair $(GR_n(F),\circ)$, consisting of a set of all invertible rhotrices of size n having entries from an arbitrary field F and together with the binary operation of row-column method for rhotrix multiplication that forms a group of all non-singular rhotrices of size n, which we term as '*the non-commutative general rhotrix group*. We identify certain subgroups of $(GR_n(F),\circ)$ and then proceed to show that its particular subgroup is embedded in a particular subgroup of the well-known general linear group. In the process, a number of theorems will be developed.

2 Definitions

The following definition will serve in our discussion.

Definition 2.1. A rhotrix R_n is said to be **invertible** or non-singular if the determinant is non-zero. That is R_n is invertible iff $\det(R_n) \neq 0$.

Definition 2.2. Set of all invertible rhotrices of size *n* is a collection of all rhotrices of size n with entries from a field F and satisfying the property that the determinant of all the rhotrices is non-zero. We denote such collection as $GR_n(F)$. Thus,

$$GR_n(F)=\left\{\left\langle\begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ a_{t1} & \ldots & \ldots & \ldots & \ldots & \ldots & a_{1t} \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{array}\right\rangle : a_{ij}, c_{lk} \in F \text{ and } \det([a_{ij}]) \neq 0 \neq \det([c_{lk}])\right\},$$

where $1 \le i, j \le t,\ 1 \le l, k \le t-1;\ t = \dfrac{n+1}{2}\ and\ n \in 2Z^{+} + 1.$

3 The general non-commutative rhotrix group

In [4], it was noted that the set of all invertible rhotrices of size *n* with entries from the set of real numbers is a group with respect to row-column method for rhotrix multiplication. We generalize this notion in the following theorem.

Theorem 3.1 (A generalization of non-commutative rhotrix groups). Let $GR_n(F)$ be the set of all invertible rhotrices with entries from an arbitrary field *F* and let $\circ$ be the row-column method for rhotrix multiplication. Then, the pair $(GR_n(F), \circ)$ is a non-commutative general rhotrix group of size *n* over *F*.

Proof: We shall show that the pair $(GR_n(F), \circ)$ is a group under the binary operation of row-column multiplication of rhotrices, i.e., we shall show that the following group axioms are satisfied:

(i) Closure: For any two rhotrices of $A_n, B_n \in GR_n(F), \det(A_n) \neq 0 \Rightarrow A_n$ is invertible, and $\det(B_n) \neq 0 \Rightarrow B_n$ is invertible.
Now, $A_n \circ B_n \in GR_n(F)$ since $\det(A_n \circ B_n) = \det(A_n).\det(B_n) \neq 0.$
Thus, $GR_n(F)$ is closed under the group binary operation.

(ii) Associativity: For all $A_n, B_n\ and\ C_n \in GR_n(F),\ (A_n \circ B_n) \circ C_n = A_n \circ (B_n \circ C_n).$

(iii) Existence of identity: For each $R_n \in GR_n(F), \exists$

$$I_n = \left\langle I_{t\times t}, I_{(t-1)\times(t-1)}\right\rangle = \left\langle\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0 & \ldots & \ldots & 1 & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array}\right\rangle \in GR_n(F),$$

such that $I_n \circ R_n = R_n \circ I_n = R_n$

(iv) Existence of inverse: For each $A_n \in GR_n(F)$, $\exists A_n^{-1} \in GR_n(F)$ such that $A_n \circ A_n^{-1} = I_n \in GR_n(F)$.

Hence, the pair $(GR_n(F), \circ)$ is a non-commutative general rhotrix group of size *n* over *F*. □

Corollary 3.2. Let $GR_n(\Re)$ be the set of all invertible rhotrices of size *n* with entries in $\Re$. Let $\circ$ be the row-column multiplication of rhotrices; then the pair $(GR_n(\Re), \circ)$ is the general non-commutative group of all invertible real rhotrices of size *n*.

Proof: By substituting $F = \Re$ in Theorem 3.1 above, the result follows. □

Corollary 3.3. The pair $(GR_3(\Re), \circ)$ is a general non-commutative group of all invertible real rhotrices of size 3.

Proof: Putting $F = \Re$ and $n = 3$ in Theorem 3.1, it follows that the pair $(GR_3(\Re), \circ)$ is the general non-commutative group of all invertible real rhotrices of size 3. This completes the proof. □

Theorem 3.4. The non-commutative general rhotrix group $(GR_n(F), \circ)$ is embedded in the general linear group $\left(GL_n(F), \cdot\right)$.

Proof: Let $(GR_n(F), \circ)$ be a group of all invertible *n*-dimensional rhotrices and let $\left(GL_n(F), \cdot\right)$ be the group of all invertible matrices of dimension *n x n*. We define the mapping $\theta : \left((GR_n(F), \circ)\right) \rightarrow \left(GL_n(F), \cdot\right)$ by

$$\theta\left(\left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle\right) = \begin{bmatrix} a_{11} & 0 & a_{12} & 0 & \cdots & \cdots & \cdots & 0 & a_{1t} \\ 0 & c_{11} & 0 & c_{12} & \cdots & \cdots & \cdots & c_{1(t-1)} & 0 \\ a_{21} & 0 & a_{22} & 0 & \cdots & \cdots & \cdots & 0 & a_{11} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(t-1)1} & 0 & a_{(t-1)2} & & \cdots & \cdots & \cdots & 0 & a_{(t-1)t} \\ 0 & c_{(t-1)1} & 0 & c_{(t-1)2} & \cdots & \cdots & \cdots & c_{(t-1)(t-1)} & 0 \\ a_{t1} & 0 & a_{t2} & 0 & \cdots & \cdots & \cdots & 0 & a_{tt} \end{bmatrix}$$

where θ maps each R_n in $GR_n(F)$ to its corresponding filled coupled matrix M_n in $\left(GL_n(F), \cdot\right)$. The operations $\circ$ and $\cdot$ denote respectively, the row-column based method for multiplication of rhotrices and the usually matrix multiplication.

Proof: Clearly, θ is a $1-1$ homomorphism since no two different rhotrices may have the same filled coupled matrix, hence $(GR_n(F), \circ)$ is embedded in $\left(GL_n(F), \cdot\right)$. This completes the proof. □

4 Subgroups of generalized non-commutative rhotrix group

In what follows, various types of rhotrices are identifying to form elements of the underlying set of the subgroups of the general non-commutative rhotrix group.

Definition (Unitary Rhotrices): A rhotrix R_n is called a unitary rhotrix if the determinant of R_n is equal to 1. We denote the set of all such unitary rhotrices of size n with entries from field F as $SR_n(F)$. Thus,

$$SR_n(F)=\left\{\left\langle \begin{array}{ccccccccccc} & & & & & a_{11} & & & & & \\ & & & & a_{21} & c_{11} & a_{12} & & & & \\ & & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & & \\ & & & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & & & \\ & & & & & a_{tt} & & & & & \end{array}\right\rangle : a_{ij}, c_{lk}\in F, \det([a_{ij}])=1=\det([c_{lk}])\right\}.$$

Theorem 4.1. (Special rhotrix subgroup) The pair $(SR_n(F),\circ)$ is a special rhotrix subgroup of $(GR_n(F),\circ)$.

Proof: Since $I_n\in SR_n(F)$, then $SR_n(F)\neq\varnothing$. Now, Let A_n and $B_n \in SR_n(F)$. Then it follows that, $\det(A_n)=1\neq 0$ and $\det(B_n)=1\neq 0$, respectively. This implies that for each A_n and $B_n\in SR_n(F)$, $\exists\ A_n^{-1}$ and $B_n^{-1}\in SR_n(F)\ni A_n\circ B_n^{-1}\in SR_n(F)$ and
$\det(A_n\circ B_n^{-1})=\det(A_n)\circ\det(B_n^{-1})=1\circ 1^{-1}=1.$

Hence $(SR_n(F),\circ)$ is a subgroup of $(GR_n(F),\circ)$. □

Theorem 4.2. The special rhotrix subgroup $(SR_n(F),\circ)$ of $(GR_n(F),\circ)$ is a embedded in the special linear subgroup $(SL_n(F),\cdot)$ of $(GL_n(F),\cdot)$.

Proof: Let $(SR_n(F),\circ)$ be a special rhotrix group of $(GR_n(F),\circ)$ and let $(SL_n(F),\cdot)$ be a special linear subgroup of $(GL_n(F),\cdot)$ We define a mapping $\theta:(SR_n(F),\circ)\to(SL_n(F),\cdot)$ by

$$\theta\left(\left\langle \begin{array}{ccccccccccc} & & & & & a_{11} & & & & & \\ & & & & a_{21} & c_{11} & a_{12} & & & & \\ & & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & & \\ & & & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & & & \\ & & & & & a_{tt} & & & & & \end{array}\right\rangle\right)$$

$$
=\begin{bmatrix}
a_{11} & 0 & a_{12} & 0 & \cdots & \cdots & \cdots & 0 & a_{1t} \\
0 & c_{11} & 0 & c_{12} & \cdots & \cdots & \cdots & c_{1(t-1)} & 0 \\
a_{21} & 0 & a_{22} & 0 & \cdots & \cdots & \cdots & 0 & a_{11} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{(t-1)1} & 0 & a_{(t-1)2} & & \cdots & \cdots & \cdots & 0 & a_{(t-1)t} \\
0 & c_{(t-1)1} & 0 & c_{(t-1)2} & \cdots & \cdots & \cdots & c_{(t-1)(t-1)} & 0 \\
a_{t1} & 0 & a_{t2} & 0 & \cdots & \cdots & \cdots & 0 & a_{tt}
\end{bmatrix},
$$

where θ maps each R_n in $SR_n(F)$ to its corresponding filled coupled matrix M_n in $\left(SL_n(F),\cdot\right)$. It is clear to see that θ is an injective homomorphism. Furthermore,

$$
\det\left(\left\langle \begin{array}{ccccccccc}
 & & & & a_{11} & & & & \\
 & & & a_{21} & c_{11} & a_{12} & & & \\
 & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
 & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\
 & & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\
 & & & & a_{tt} & & & &
\end{array}\right\rangle\right)
$$

$$
=\det\left(\begin{bmatrix}
a_{11} & 0 & a_{12} & 0 & \cdots & \cdots & \cdots & 0 & a_{1t} \\
0 & c_{11} & 0 & c_{12} & \cdots & \cdots & \cdots & c_{1(t-1)} & 0 \\
a_{21} & 0 & a_{22} & 0 & \cdots & \cdots & \cdots & 0 & a_{11} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{(t-1)1} & 0 & a_{(t-1)2} & & \cdots & \cdots & \cdots & 0 & a_{(t-1)t} \\
0 & c_{(t-1)1} & 0 & c_{(t-1)2} & \cdots & \cdots & \cdots & c_{(t-1)(t-1)} & 0 \\
a_{t1} & 0 & a_{t2} & 0 & \cdots & \cdots & \cdots & 0 & a_{tt}
\end{bmatrix}\right)=1
$$

Hence the result follows. □

Definition (Diagonal Rhotrix): A rhotrix R_n is called a diagonal rhotrix if all the elements in the vertical diagonal are non-zero, while others are zeros. We denote the set of all invertible diagonal rhotrices of size n as $DR_n(F)$. Thus,

$$DR_n(F)=\left\{\left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & 0 & & & \\ & & 0 & 0 & a_{22} & 0 & 0 & & \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & c_{(t-1)(t-1)} & 0 & & & \\ & & & & a_{tt} & & & & \end{array}\right\rangle : a_{ij}, c_{lk} \in F, \det([a_{ij}]) \neq 0 \neq \det([c_{lk}])\right\}.$$

Theorem 4.3. (Diagonal rhotrix subgroup) The pair $(DR_n(F),\circ)$ is a rhotrix subgroup of $(GR_n(F),\circ)$.

Proof: $DR_n(F) \neq \varnothing$ since

$$I_n=\left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \dots & \dots. & \dots & \dots & \dots & \\ 0 & \dots & \dots & 1 & \dots & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array}\right\rangle \in DR_n(F).$$

Next, let

$$A_n=(p_{ij},q_{lk})=\left\langle \begin{array}{ccccccccc} & & & & p_{11} & & & & \\ & & & 0 & q_{11} & 0 & & & \\ & & 0 & 0 & p_{22} & 0 & 0 & & \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ & & 0 & 0 & p_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & q_{(t-1)(t-1)} & 0 & & & \\ & & & & p_{tt} & & & & \end{array}\right\rangle \in DR_n(F)$$

and

$$B_n=(r_{ij},s_{lk})=\left\langle \begin{array}{ccccccccc} & & & & r_{11} & & & & \\ & & & 0 & s_{11} & 0 & & & \\ & & 0 & 0 & r_{22} & 0 & 0 & & \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ & & 0 & 0 & r_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & s_{(t-1)(t-1)} & 0 & & & \\ & & & & r_{tt} & & & & \end{array}\right\rangle \in DR_n(F)$$

it follows that $\det(A_n) \neq 0$ and $\det(B_n) \neq 0$ respectively. Implying that A_n^{-1} and B_n^{-1} exist in $DR_n(F)$. So,

$$A_n \circ B_n^{-1} = \left\langle \begin{array}{c} p_{11} \\ 0 \quad q_{11} \quad 0 \\ 0 \quad 0 \quad p_{22} \quad 0 \quad 0 \\ \cdots \; \cdots \; \cdots \quad \cdots \quad \cdots \; \cdots \\ 0 \quad \cdots \; \cdots \; \cdots \quad \cdots \quad \cdots \; \cdots \; \cdots \quad 0 \\ \cdots \; \cdots \; \cdots \quad \cdots \quad \cdots \; \cdots \\ 0 \quad 0 \quad p_{(t-1)(t-1)} \quad 0 \quad 0 \\ 0 \quad q_{(t-1)(t-1)} \quad 0 \\ p_{tt} \end{array} \right\rangle \circ \left\langle \begin{array}{c} \frac{1}{r_{11}} \\ 0 \quad \frac{1}{s_{11}} \quad 0 \\ 0 \quad 0 \quad \frac{1}{r_{22}} \quad 0 \quad 0 \\ \cdots \; \cdots \; \cdots \quad \cdots \quad \cdots \; \cdots \\ 0 \quad \cdots \; \cdots \; \cdots \quad \cdots \quad \cdots \; \cdots \; \cdots \quad 0 \\ \cdots \; \cdots \; \cdots \quad \cdots \quad \cdots \; \cdots \\ 0 \quad 0 \quad \frac{1}{r_{(t-1)(t-1)}} \quad 0 \quad 0 \\ 0 \quad \frac{1}{s_{(t-1)(t-1)}} \quad 0 \\ \frac{1}{r_{tt}} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} \frac{p_{11}}{r_{11}} \\ 0 \quad \frac{q_{11}}{s_{11}} \quad 0 \\ 0 \quad 0 \quad \frac{p_{22}}{r_{221}} \quad 0 \quad 0 \\ \cdots \; \cdots \; \cdots \quad \cdots \quad \cdots \; \cdots \\ 0 \quad \cdots \; \cdots \; \cdots \quad \cdots \quad \cdots \; \cdots \; \cdots \quad 0 \\ \cdots \; \cdots \; \cdots \quad \cdots \quad \cdots \; \cdots \\ 0 \quad 0 \quad \frac{p_{(t-1)(t-1)}}{r_{(t-1)(t-1)}} \quad 0 \quad 0 \\ 0 \quad \frac{q_{(t-1)(t-1)}}{s_{(t-1)(t-1)}} \quad 0 \\ \frac{p_{tt}}{r_{tt}} \end{array} \right\rangle \in DR_n(F)$$

Hence $(DR_n(F), \circ)$ is a rhotrix subgroup of $(GR_n(F), \circ)$. □

Theorem 4.4. The Diagonal rhotrix subgroup $(DR_n(F), \circ)$ of $(GR_n(F), \circ)$ is embedded in the diagonal linear subgroup $(DL_n(F), \cdot)$ of $(GL_n(F), \cdot)$.

Proof: Let $(DR_n(F), \circ)$ be a diagonal rhotrix subgroup of $(GR_n(F), \circ)$ and let $(DL_n(F), \cdot)$ be a diagonal linear subgroup of $(GL_n(F), \cdot)$. We define a mapping $\phi : (DR_n(F), \circ) \rightarrow (DL_n(F), \cdot)$ by

$$\phi\left(\left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & 0 & & & \\ & & 0 & 0 & a_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & 0 & 0 & 0 & & \\ & & & 0 & c_{(t-1)(t-1)} & 0 & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle\right) = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & c_{11} & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & a_{22} & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & c_{(t-1)(t-1)} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & a_{tt} \end{bmatrix}$$

where ϕ mapped each rhotrix R_n in $DR_n(F)$, to its filled coupled matrix M_n in $DL_n(F)$. Clearly, ϕ is an injective homomorphism. Since no two rhotrices have the same filled coupled matrix, hence the diagonal rhotrix subgroup is embedded in the diagonal linear subgroup. ☐

Definition (Scalar Rhotrix): A rhotrix R_n is called a scalar rhotrix if all the elements in the vertical diagonal are non-zero scalar, while others are zero(s). Scalar rhotrices are rhotrices of the form KI, where I is the identity rhotrix and K is a non-zero constant.

We denote the set of all invertible scalar rhotrices of size n as $KR_n(F)$. Thus,

$$KR_n(F) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & k_{11} & & & & \\ & & & 0 & \kappa_{11} & 0 & & & \\ & & 0 & 0 & k_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & k_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & \kappa_{(t-1)(t-1)} & 0 & & & \\ & & & & k_{tt} & & & & \end{array} \right\rangle : a_{ij}, c_{lk} \in F, \det([a_{ij}]) \neq 0 \neq \det([c_{lk}]) \right\}.$$

Theorem 4.5. (Scalar rhotrix subgroup) The pair $(KR_n(F), \circ)$ is a rhotrix subgroup of $(GR_n(F), \circ)$.

Proof: $KR_n(F) \neq \varnothing$ since

$$I_n = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & \cdots & 1 & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle \in KR_n(F).$$

Next, let

$$
A_n = \left\langle p_{ij}, p_{lk} \right\rangle = \left\langle \begin{array}{ccccccccc}
 & & & & p_{11} & & & & \\
 & & & 0 & p_{11} & 0 & & & \\
 & & 0 & 0 & p_{11} & 0 & 0 & & \\
 & \dots & \dots & \dots & \dots & \dots & \dots & & \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 & \dots & \dots & \dots & \dots & \dots & \dots & & \\
 & & 0 & 0 & p_{11} & 0 & 0 & & \\
 & & & 0 & p_{11} & 0 & & & \\
 & & & & p_{11} & & & &
\end{array} \right\rangle \in KR_n(F)
$$

and

$$
B_n = \left\langle r_{ij}, r_{lk} \right\rangle = \left\langle \begin{array}{ccccccccc}
 & & & & r_{11} & & & & \\
 & & & 0 & r_{11} & 0 & & & \\
 & & 0 & 0 & r_{11} & 0 & 0 & & \\
 & \dots & \dots & \dots & \dots & \dots & \dots & & \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 & \dots & \dots & \dots & \dots & \dots & \dots & & \\
 & & 0 & 0 & r_{11} & 0 & 0 & & \\
 & & & 0 & r_{11} & 0 & & & \\
 & & & & r_{11} & & & &
\end{array} \right\rangle \in KR_n(F).
$$

It follows that $\det(A_n) \neq 0$ and $\det(B_n) \neq 0$ respectively. Implying that A_n^{-1} and B_n^{-1} exist in $KR_n(F)$. So,

$$
A_n \circ B_n^{-1} = \left\langle \begin{array}{ccccccccc}
 & & & & p_{11} & & & & \\
 & & & 0 & p_{11} & 0 & & & \\
 & & 0 & 0 & p_{11} & 0 & 0 & & \\
 & \dots & \dots & \dots & \dots & \dots & \dots & & \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 & \dots & \dots & \dots & \dots & \dots & \dots & & \\
 & & 0 & 0 & p_{11} & 0 & 0 & & \\
 & & & 0 & p_{11} & 0 & & & \\
 & & & & p_{11} & & & &
\end{array} \right\rangle \circ \left\langle \begin{array}{ccccccccc}
 & & & & \frac{1}{r_{11}} & & & & \\
 & & & 0 & \frac{1}{r_{11}} & 0 & & & \\
 & & 0 & 0 & \frac{1}{r_{11}} & 0 & 0 & & \\
 & \dots & \dots & \dots & \dots & \dots & \dots & & \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 & \dots & \dots & \dots & \dots & \dots & \dots & & \\
 & & 0 & 0 & \frac{1}{r_{11}} & 0 & 0 & & \\
 & & & 0 & \frac{1}{r_{11}} & 0 & & & \\
 & & & & \frac{1}{r_{11}} & & & &
\end{array} \right\rangle
$$

$$
= \left\langle \begin{array}{c}
\frac{p_{11}}{r_{11}} \\
0 \quad \frac{p_{11}}{r_{11}} \quad 0 \\
0 \quad 0 \quad \frac{p_{11}}{r_{11}} \quad 0 \quad 0 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
0 \quad 0 \quad \frac{p_{11}}{r_{11}} \quad 0 \quad 0 \\
0 \quad \frac{p_{11}}{r_{11}} \quad 0 \\
\frac{p_{11}}{r_{11}}
\end{array} \right\rangle \in KR_n(F)
$$

Hence $(KR_n(F), \circ)$ is a scalar rhotrix subgroup of $(GR_n(F), \circ)$. □

Theorem 4.6. The scalar rhotrix subgroup $(KR_n(F), \circ)$ of $(GR_n(F), \circ)$ is embedded in the Scalar linear subgroup $(KL_n(F), \cdot)$ of $(GL_n(F), \cdot)$.

Proof: Let $(KR_n(F), \circ)$ be a scalar rhotrix subgroup of $(GR_n(F), \circ)$ and let $(KL_n(F), \cdot)$ be a scalar linear subgroup of $(GL_n(F), \cdot)$. We define a mapping $\mu : (KR_n(F), \circ) \rightarrow (KL_n(F), \cdot)$ by

$$
\mu \left(\left\langle \begin{array}{c}
a_{11} \\
0 \quad a_{11} \quad 0 \\
0 \quad 0 \quad a_{11} \quad 0 \quad 0 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad a_{11} \quad 0 \\
a_{11}
\end{array} \right\rangle \right) = \begin{bmatrix}
a_{11} & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\
0 & a_{11} & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\
0 & 0 & a_{11} & 0 & \cdots & \cdots & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & a_{11} & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & a_{11}
\end{bmatrix},
$$

where μ maps each rhotrix R_n in $KR_n(F)$, to its filled coupled matrix M_n in $KL_n(F)$, clearly, it follows that:

$$\mu(A_n \circ B_n) = \mu(A_n) \cdot \mu(B_n) \ \forall \ A_n, B_n \in GR_n(F)$$

μ is a homomorphism. Also, μ is 1 – 1 since no two rhotrices have the same filled coupled matrix. □

Definition (Left triangular rhotrix): A rhotrix R_n is called a left triangular rhotrix if all the elements in the right of the vertical diagonal are all zero.

We denote the set of all invertible left triangular rhotrices of size n as $LTR_n(F)$. Thus,

$$
LTR_n(F)=\left\{\left\langle\begin{array}{c}
a_{11}\\
a_{21}\quad c_{11}\quad 0\\
a_{31}\quad c_{21}\quad a_{22}\quad 0\quad 0\\
\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
a_{t1}\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad 0\\
\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
a_{t(t-2)}\quad c_{(t-1)(t-2)}\quad a_{(t-1)(t-1)}\quad 0\quad 0\\
a_{t(t-1)}\quad c_{(t-1)(t-1)}\quad 0\\
a_{tt}
\end{array}\right\rangle : a_{ij}, c_{lk}\in F, \det(a_{ij})\neq 0\neq \det(c_{lk}),\right\}.
$$

$$
where\ \ a_{ij}=0\ \ if\ \ i<j\quad and\quad c_{lk}=0\ if\ \ l<k
$$

Theorem 4.7. (Left triangular rhotrix subgroup) The pair $(LTR_n(F),\circ)$ is a rhotrix subgroup of $(GR_n(F),\circ)$.

Proof: Since
$$
I_n=\left\langle\begin{array}{c}
1\\
0\quad 1\quad 0\\
\cdots\quad\cdots.\quad\cdots\quad\cdots\quad\cdots\\
0\quad\cdots\quad\cdots\quad 1\quad\cdots\quad\cdots\quad 0\\
\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
0\quad 1\quad 0\\
1
\end{array}\right\rangle\in LTR_n(F),\ \text{then}\ LTR_n(F)\neq\varnothing.
$$

Let

$$
A_n=\langle a_{ij}, c_{lk}\rangle=\left\langle\begin{array}{c}
a_{11}\\
a_{21}\quad c_{11}\quad 0\\
a_{31}\quad c_{21}\quad a_{22}\quad 0\quad 0\\
\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
a_{t1}\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad 0\\
\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
a_{t(t-2)}\quad c_{(t-1)(t-2)}\quad a_{(t-1)(t-1)}\quad 0\quad 0\\
a_{t(t-1)}\quad c_{(t-1)(t-1)}\quad 0\\
a_{tt}
\end{array}\right\rangle
$$

and

$$
B_n=\langle b_{ij}, d_{lk}\rangle=\left\langle\begin{array}{c}
b_{11}\\
b_{21}\quad d_{11}\quad 0\\
b_{31}\quad d_{21}\quad b_{22}\quad 0\quad 0\\
\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
b_{t1}\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad 0\\
\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
b_{t(t-2)}\quad d_{(t-1)(t-2)}\quad b_{(t-1)(t-1)}\quad 0\quad 0\\
b_{t(t-1)}\quad d_{(t-1)(t-1)}\quad 0\\
b_{tt}
\end{array}\right\rangle
$$

be two rhotrices of size n in $LTR_n(F)$, it follows that $(A_n \circ B_n) \in LTR_n(F)$, since the product of two left triangular rhotrices is also a left triangular rhotrix.

So the set $LTR_n(F)$ is closed under the operation of rhotrix multiplication. Next, for any $A_n \in LTR_n(F)$, $A_n^{-1} \in LTR_n(F)$ since $\det(A_n) \neq 0$.

Now we have $(A_n \circ B_n^{-1}) \in LTR_n(F) \forall\ A_n, B_n \in LTR_n(F)$.

Hence $(LTR_n(F), \circ)$ is a subgroup of $(GR_n(F), \circ)$. ☐

Theorem 4.8. Let $(LTR_n(F), \circ)$ be the left triangular rhotrix subgroup of $(GR_n(F), \circ)$ and let $(LTL_n(F), \cdot)$ be the lower triangular linear subgroup of $(GL_n(F), \cdot)$ then $(LTR_n(F), \circ)$ is embedded in $(LTL_n(F), \cdot)$.

Proof: Let $(LTR_n(F), \circ)$ be a Left triangular rhotrix subgroup and let $(LTL_n(F), \cdot)$ lower triangular linear subgroup. We define a mapping $\varphi : (LTR_n(F), \circ) \to (LTL_n(F), \cdot)$ by

$$\varphi\left(\left\langle \begin{array}{c} a_{11} \\ a_{21} \quad c_{11} \quad 0 \\ a_{31} \quad c_{21} \quad a_{22} \quad 0 \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{t1} \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{t(t-2)} \quad c_{(t-1)(t-2)} \quad a_{(t-1)(t-1)} \quad 0 \quad 0 \\ a_{t(t-1)} \quad c_{(t-1)(t-1)} \quad 0 \\ a_{tt} \end{array} \right\rangle\right) = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & c_{11} & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ a_{21} & 0 & a_{22} & 0 & \cdots & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(t-1)1} & 0 & a_{(t-1)2} & 0 & \cdots & \cdots & a_{(t-1)(t-1)} & 0 & 0 \\ 0 & c_{(t-1)1} & 0 & c_{(t-1)2} & \cdots & \cdots & 0 & c_{(t-1)(t-1)} & 0 \\ a_{t1} & 0 & a_{t2} & 0 & \cdots & \cdots & a_{t(t-1)} & 0 & a_{tt} \end{bmatrix}$$

Where φ maps every left triangular rhotrix to its correspondence filled coupled lower triangular matrix. We observe that φ is an injective homomorphism Hence, the left triangular rhotrix subgroup is embedded in the left triangular matrix group. ☐

Definition (Special left triangular rhotrix): A rhotrix R_n is called a special left triangular rhotrix if all the elements in the right of the vertical diagonal are all zero and $\det(R_n) = 1$.

We denote the set of all special left triangular rhotrices of size n as $SLTR_n(F)$. Thus,

$$SLTR_n(F) = \left\{ \begin{array}{l} \left\langle \begin{array}{c} a_{11} \\ a_{21} \quad c_{11} \quad 0 \\ a_{31} \quad c_{21} \quad a_{22} \quad 0 \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{t1} \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{t(t-2)} \quad c_{(t-1)(t-2)} \quad a_{(t-1)(t-1)} \quad 0 \quad 0 \\ a_{t(t-1)} \quad c_{(t-1)(t-1)} \quad 0 \\ a_{tt} \end{array} \right\rangle : a_{ij}, c_{lk} \in F, \det(a_{ij}) = 1 = \det(c_{lk}) \\ where\ \ a_{ij} = 0 \ \ if\ \ i < j \ \ and \ \ c_{lk} = 0 \ if\ \ l < k \end{array} \right\}.$$

Theorem 4.9. (Special left triangular rhotrix subgroup) The pair $(SLTR_n(F), \circ)$ is a rhotrix subgroup of $(GR_n(F), \circ)$.

Proof: Since $I_n \in SLTR_n(F)$ then $SLTR_n(F) \neq \varnothing$. Now, Let A_n and $B_n \in SLTR_n(F)$. Then $\det(A_n) = 1 \neq 0$ and $\det(B_n) = 1 \neq 0$, respectively. This implies that for each $A_n, B_n \in SLTR_n(F)$ $\exists$ A_n^{-1} and $B_n^{-1} \in SLTR_n(F) \ni A_n \circ B_n^{-1} \in SLTR_n(F)$ and $\det(A_n \circ B_n^{-1})$ $= \det(A_n).\det(B_n^{-1}) = 1.1^{-1} = 1$.

Hence $SLTR_n(F)$ is a subgroup of $(GR_n(F), \circ)$. □

Definition (Right triangular rhotrix): A rhotrix R_n is called a right triangular rhotrix if all the elements in the left of the vertical diagonal are all zero. We denote the set of all invertible right triangular rhotrices of size n as $RTR_n(F)$

.

$$RTR_n(F) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & a_{12} & & & \\ & & 0 & 0 & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle : a_{ij}, c_{lk} \in F, \det(a_{ij}) \neq 0, \det(c_{lk}) \neq 0 \right\},$$

where $a_{ij} = 0$ *if* $i > j$ *and* $c_{lk} = 0$ *if* $l > k$.

Theorem 4.10. (Right triangular rhotrix subgroup) The pair $(RTR_n(F), \circ)$ is a rhotrix subgroup of $(GR_n(F), \circ)$.

Proof: Since $I_n = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & \cdots & 1 & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle \in RTR_n(F)$, then $RTR_n(F) \neq \varnothing$.

Let

$$A_n=\langle a_{ij},c_{lk}\rangle=\left\langle\begin{array}{c} a_{11}\\ 0\quad c_{11}\quad a_{12}\\ 0\quad 0\quad a_{22}\quad c_{12}\quad a_{13}\\ \cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\\ 0\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;a_{1t}\\ \cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\\ 0\quad 0\quad a_{(t-1)(t-1)}\quad c_{(t-2)(t-1)}\quad a_{(t-2)t}\\ 0\quad c_{(t-1)(t-1)}\quad a_{(t-1)t}\\ a_{tt}\end{array}\right\rangle$$

and

$$B_n=\langle b_{ij},d_{lk}\rangle=\left\langle\begin{array}{c} b_{11}\\ 0\quad d_{11}\quad b_{12}\\ 0\quad 0\quad b_{22}\quad d_{12}\quad b_{13}\\ \cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\\ 0\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;b_{1t}\\ \cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\\ 0\quad 0\quad b_{(t-1)(t-1)}\quad d_{(t-2)(t-1)}\quad b_{(t-2)t}\\ 0\quad d_{(t-1)(t-1)}\quad b_{(t-1)t}\\ b_{tt}\end{array}\right\rangle$$

be two rhotrices of size n in $RTR_n(F)$, it follows that $(A_n\circ B_n)\in RTR_n(F)$, since the product of two right triangular rhotrices is a right triangular rhotrix. So the set $RTR_n(F)$, is closed under the operation of rhotrix multiplication.

Next, for any $A_n\in RTR_n(F)$, $A_n^{-1}\in RTR_n(F)$ since $\det(A_n)\neq 0$

Now we have $(A_n\circ B_n^{-1})\in RTR_n(F)\,\forall\, A_n, B_n\in RTR_n(F)$.

Hence $(RTR_n(F),\circ)$ is a subgroup of $(GR_n(F),\circ)$. □

Theorem 4.11. Let $(RTR_n(F),\circ)$ be the right triangular rhotrix subgroup of $(GR_n(F),\circ)$ and let $(UTL_n(F),\cdot)$ be the upper triangular linear subgroup of $(GL_n(F),\cdot)$, then $(RTR_n(F),\circ)$ is embedded in $(UTL_n(F),\cdot)$.

Proof: Let $(RTR_n(F),\circ)$ be a Left triangular rhotrix subgroup and let $(UTL_n(F),\cdot)$ upper triangular matrix group. We define a mapping $\varphi:(RTR_n(F),\circ)\to(UTL_n(F),\cdot)$ by

$$\varphi\left(\left\langle\begin{array}{c} a_{11}\\ 0\quad c_{11}\quad a_{12}\\ 0\quad 0\quad a_{22}\quad c_{12}\quad a_{13}\\ \cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\\ 0\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;a_{1t}\\ \cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\\ 0\quad 0\quad a_{(t-1)(t-1)}\quad c_{(t-2)(t-1)}\quad a_{(t-2)t}\\ 0\quad c_{(t-1)(t-1)}\quad a_{(t-1)t}\\ a_{tt}\end{array}\right\rangle\right)=\begin{bmatrix} a_{11} & 0 & a_{12} & 0 & \cdots & \cdots & a_{1(t-1)} & 0 & a_{1t}\\ 0 & c_{11} & 0 & c_{12} & \cdots & \cdots & 0 & c_{1(t-1)} & 0\\ 0 & 0 & a_{22} & 0 & \cdots & \cdots & a_{2(t-1)} & 0 & a_{2t}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots. & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & \cdots & \cdots & a_{(t-1)(t-1)} & 0 & a_{(t-1)t}\\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & c_{(t-1)(t-1)} & 0\\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & a_{tt}\end{bmatrix},$$

where φ maps every right triangular rhotrix to its correspondence filled coupled upper triangular matrix. We observe that φ is an injective homomorphism hence the right triangular rhotrix group is embedded in the upper triangular matrix group. □

Definition (Special right triangular rhotrix): A rhotrix R_n is called a special right triangular rhotrix if all the elements in the left of the vertical diagonal are all zero and $\det(R_n)=1$.
We denote the set of all special right triangular rhotrices of size n as $SRTR_n(F)$. Thus,

$$SRTR_n(F)=\left\{\left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & a_{12} & & & \\ & & 0 & 0 & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array}\right\rangle : a_{ij}, c_{lk}\in F, \det(a_{ij})=1, \det(c_{lk})=1\right\},$$

where $a_{ij}=0 \;\; if \;\; i>j \;\; and \;\; c_{lk}=0 \; if \;\; l>k$.

Theorem 4.12. (Special right triangular rhotrix subgroup) The pair $(SRTR_n(F),\circ)$ is a rhotrix subgroup of $(GR_n(F),\circ)$.

Proof: Since $I_n\in SRTR_n(F)$ then $SRTR_n(F)\neq\varnothing$. Now, Let A_n and $B_n \in SRTR_n(F)$. Then *det(* A_n *)* $=1\neq 0$ and *det(* B_n) $=1\neq 0$, respectively. This implies that for each A_n and $B_n\in SRTR_n(F)$ $\exists\, A_n^{-1}$ and $B_n^{-1}\in SRTR_n(F) \ni A_n\circ B_n^{-1}\in SRTR_n(F)$ and $\det(A_n\circ B_n^{-1})$ $=\det(A_n).\det(B_n^{-1})=1.1^{-1}=1$. Hence, $SRTR_n(F)$ is a subgroup of $(GR_n(F),\circ)$. □

5 Isomorphism between some subgroups of the general non-commutative rhotrix group

The following theorem establishes an isomorphic relationship between left triangular rhotrix group and right triangular rhotrix group.

Theorem 5.1. Let φ be a mapping from $(LTR_n(F),\circ)$ to $(RTR_n(F),\circ)$ defined by

$$\varphi\left(\left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & \\ & & & a_{tt} & & & \end{array}\right\rangle\right)=\left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & 0 & c_{11} & a_{12} & & \\ & & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & & \cdots & \cdots & \cdots & & \\ & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{array}\right\rangle$$

Then the mapping φ is an isomorphism.

Proof: Let $(LTR_n(F),\circ)$ and $(RTR_n(F),\circ)$ be the group of all left triangular rhotrices of size n and the group of all right triangular rhotrices of size n respectively, we define a mapping

$$\varphi:(LTR_n(F),\circ)\to(RTR_n(F),\circ)$$

by

$$\varphi(R_n)=\varphi\left(\left\langle a_{ij},c_{lk}\right\rangle\right)=\left\langle a_{ji},c_{kl}\right\rangle.$$

This is a homomorphism since if $R_n=\left\langle a_{i_1j_1},c_{l_1k_1}\right\rangle$ and $Q_n=\left\langle b_{i_2j_2},d_{l_2k_2}\right\rangle$ then

$$\begin{aligned}
\varphi(R_n\circ Q_n)&=\varphi\left(\left\langle a_{i_1j_1},c_{l_1k_1}\right\rangle\circ\left\langle b_{i_2j_2},d_{l_2k_2}\right\rangle\right)\\
&=\varphi\left(\sum_{i_2j_1=1}^{t}a_{i_1j_1}b_{i_2j_2},\sum_{l_2k_1=1}^{t-1}c_{l_1k_1}d_{l_2k_2}\right)\\
&=\left(\sum_{i_2j_1=1}^{t}a_{j_1i_1}b_{j_2i_2},\sum_{l_2k_1=1}^{t-1}c_{k_1l_1}d_{k_2l_2}\right)\\
&=\left\langle a_{j_1i_1},c_{k_1l_1}\right\rangle\circ\left\langle b_{j_2i_2},d_{k_2l_2}\right\rangle\\
&=\varphi\left(\left\langle a_{i_1j_1},c_{l_1k_1}\right\rangle\right)\circ\varphi\left(\left\langle b_{i_2j_2},d_{l_2k_2}\right\rangle\right)\\
&=\varphi(R_n)\circ\varphi(Q_n).
\end{aligned}$$

Next, φ is a bijection since $\ker(\varphi)=\left\{I_n\in(LTR_n(F),\circ):\varphi(I_n)=I_n^T\in(RTR_n(F),\circ)\right\}$. □

6 Conclusion

We have presented an algebraic study of non-commutative rhotrix groups and their generalization as non-commutative general rhotrix group $(GR_n(F),\circ)$. Many particular subgroups of $(GR_n(F),\circ)$ were identified and also shown to be embedded in a particular subgroup of the well known general linear group. Furthermore, we investigated some isomorphic relationship between some subgroups of $(GR_n(F),\circ)$. In the future, it may be interesting to consider a number of topics on non-commutative rhotrix groups, such as computing non-commutative finite groups of rhotrices, development of non-commutative finite cyclic groups of rhotrices, as well as construction of composition series for non-commutative finite groups of rhotrices.

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