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A note on a family of alternating Fibonacci sums

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Abstract: In this note we consider a family of finite and infinite alternating sums containing products of Fibonacci numbers. We derive closed-form expressions for this family of sums. As a consequence of this result we establish new algebraic relationships between certain alternating sums of reciprocals of products of Fibonacci numbers with integer power.
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1 Introduction

The Fibonacci numbers F_n are defined for $n \geq 0$ as $F_{n+2} = F_{n+1} + F_n$ with initial conditions $F_0 = 0, F_1 = 1$. The Binet form is given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad n \geq 0, \tag{1.1}$$

where α and β are roots of the quadratic equation $x^2 - x - 1 = 0$, i.e. $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ (see [7]).

The goal of this study is to evaluate finite and infinite alternating sums of Fibonacci numbers. Special attention is paid to sums of reciprocals of products of these numbers. The interest is not new and the topic is treated in several articles. Brousseau [3] derives among others the following results dealing with sums of order 1 and 2

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+2}}, \tag{1.2}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{F_nF_{n+k}}=\frac{1}{F_k}\Big(\frac{k}{\alpha}-\sum_{j=1}^{k}\frac{F_{j-1}}{F_j}\Big). \tag{1.3}$$

Series of the character of (1.2) and (1.3) are also discussed in a more general setting by Carlitz [4], Horadam [8], André-Jeannin [1, 2] and Melham [9, 10]. For instance, it is proofed in [1] that

$$\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}}=-\frac{1}{\alpha}+2\sum_{n=1}^{\infty}\frac{1}{\alpha^nF_n}. \tag{1.4}$$

Furthermore, the author of [1] shows how the last sum can be expressed in terms of the Lambert series. Brousseau [3], Carlitz [4] and Melham [9, 10] extend the study and consider sums of the form

$$\sum_{n=1}^{\infty}\frac{(\pm 1)^{n-1}}{F_nF_{n+k_1}F_{n+k_2}...F_{n+k_m}},$$

where $k_i, i = 1, ..., m$ are positive integers. They derive reduction formulas for these sums. Brousseau [3], Carlitz [4] and Good [6] have found expressions in the finite case, too. For instance, for $k \geq 1$ it holds that

$$\sum_{n=1}^{N}\frac{(-1)^n}{F_nF_{n+k}}=\frac{1}{F_k}\Big(\sum_{j=1}^{k}\frac{F_{j-1}}{F_j}-\sum_{j=1}^{k}\frac{F_{j+N-1}}{F_{j+N}}\Big) \tag{1.5}$$

or

$$\sum_{n=1}^{N}\frac{(-1)^n}{F_nF_{n+k}}=\frac{F_N}{F_k}\sum_{n=1}^{k}\frac{(-1)^n}{F_nF_{n+N}}. \tag{1.6}$$

Finally, we refer the reader to Rabinowitz [11] who discusses the general problem of evaluating Fibonacci products of reciprocals and shows that the summations may be reduced to evaluations of some basic forms. He also poses challenging open questions.

2 The results

The following Proposition will be essential for our study:

Proposition 2.1. *For integers $p, l, k \geq 0$ and $N \geq p$ it holds that*

$$F_{p+l}F_{N+1+l+k}-F_{p+l+k}F_{N+1+l}=(-1)^{p+l+1}F_kF_{N+1-p}. \tag{2.1}$$

Also, for $p, l, k \geq 0$ it holds that

$$F_{p+l}\alpha^k-F_{p+l+k}=(-1)^{p+l+1}\frac{F_k}{\alpha^{p+l}}. \tag{2.2}$$

Proof: Set $a = p + l$ and $b = N + 1 + l$ $(a \geq 0, b \geq 1)$. Then the first equation of the Proposition is equivalent to

$$F_aF_{b+k}-F_{a+k}F_b=(-1)^{a+1}F_kF_{b-a}, \quad b-a\geq 1. \tag{2.3}$$

This is a special form of equation (3.1) of Udrea [12], where a proof based on Chebyshev polynomials is given. To prove the second statement we note that

$$F_{p+l}\frac{F_{N+1+l+k}}{F_{N+1+l}} - F_{p+l+k} = (-1)^{p+l+1}F_k\frac{F_{N+1-p}}{F_{N+1+l}}. \tag{2.4}$$

Taking the limit $N \to \infty$ on both sides of the equation and using that $\lim_{n\to\infty} F_{n+m}/F_n = \alpha^m$ the desired statement follows immediately. □

The following two Corollaries will also be used in this note:

Corollary 2.2. *For integers l, k, r with $l, k \geq 0$ and $r \geq 1$ it holds that*

$$F_{n+l}^r F_{n+1+l+k}^r - F_{n+1+l}^r F_{n+l+k}^r = \sum_{i=1}^{r}\binom{r}{i}(-1)^{i(n+1+l)}F_k^i F_{n+1+l}^{r-i}F_{n+l+k}^{r-i}. \tag{2.5}$$

Proof: Let $r = 1$. Then the identity

$$F_{n+l}F_{n+1+l+k} - F_{n+1+l}F_{n+l+k} = (-1)^{n+1+l}F_k, \tag{2.6}$$

follows from the Proposition upon setting $n = p = N$. The general statement follows from the binomial identity. □

Corollary 2.3. *For $p, l, k \geq 0$, $N \geq p$ and $r \geq 1$ it holds hat*

$$F_{p+l}^r F_{N+1+l+k}^r - F_{p+l+k}^r F_{N+1+l}^r = (-1)^{p+l+1}F_k F_{N+1-p}\cdot P(F_{p+l}F_{N+1+l+k}, F_{p+l+k}F_{N+1+l}) \tag{2.7}$$

and

$$F_{p+l}^r\alpha^{kr} - F_{p+l+k}^r = (-1)^{p+l+1}\frac{F_k}{\alpha^{p+l}}\cdot P(F_{p+l}\alpha^k, F_{p+l+k}), \tag{2.8}$$

where $P(x, y)$ is a polynomial in x and y of degree $r-1$ with

$$P(x,y) = \sum_{i=1}^{r} x^{r-i}y^{i-1}. \tag{2.9}$$

Proof: For $r = 1$ the identities coincide with the statements of the Proposition. For $r > 1$ they follow from the formula

$$x^r - y^r = (x-y)(x^{r-1} + x^{r-2}y + \cdots + xy^{r-2} + y^{r-1}). \tag{2.10}$$

□

Our main results are contained in the next theorem.

Theorem 2.4. *For $l, k \geq 0$ and $r \geq 1$ define*

$$\begin{aligned} x_1 &= F_{n+l}^r F_{n+1+l+k}^r - F_{n+1+l}^r F_{n+l+k}^r \\ x_2 &= F_{n+l+k}^r F_{n+1+l+k}^r \\ x_3 &= F_{n+l}^r F_{n+1+l}^r. \end{aligned}$$

For a real number a define the sum

$$S(p,N,l,k,r,a)=\sum_{n=p}^{N}\frac{x_1(x_2-a^2x_3)}{(x_2+a^2x_3)^2+a^2x_1^2}. \tag{2.11}$$

Then

$$S(p,N,l,k,r,a)=\frac{F_{p+l}^rF_{p+l+k}^r}{F_{p+l+k}^{2r}+a^2F_{p+l}^{2r}}-\frac{F_{N+1+l}^rF_{N+1+l+k}^r}{F_{N+1+l+k}^{2r}+a^2F_{N+1+l}^{2r}}. \tag{2.12}$$

and

$$S(p,\infty,l,k,r,a)=\frac{F_{p+l}^rF_{p+l+k}^r}{F_{p+l+k}^{2r}+a^2F_{p+l}^{2r}}-\frac{\alpha^{kr}}{\alpha^{2kr}+a^2}. \tag{2.13}$$

Proof: For $a\neq 0$ and $r\geq 1$ define $h(x)=ax^r$. Further set $g(n)=F_{n+l}/F_{n+l+k}$. Then it follows from Theorem 2.1 in [5] that

$$\sum_{n=p}^{N}\tan^{-1}\left(\frac{ax_1}{x_2+a^2x_3}\right)=\tan^{-1}\left(a\frac{F_{p+l}^r}{F_{p+l+k}^r}\right)-\tan^{-1}\left(a\frac{F_{N+1+l}^r}{F_{N+1+l+k}^r}\right), \tag{2.14}$$

and

$$\sum_{n=p}^{\infty}\tan^{-1}\left(\frac{ax_1}{x_2+a^2x_3}\right)=\tan^{-1}\left(a\frac{F_{p+l}^r}{F_{p+l+k}^r}\right)-\tan^{-1}\left(\frac{a}{\alpha^{kr}}\right). \tag{2.15}$$

The stated assertions follow from differentiation w.r.t. the parameter a. □

As a direct application of the Theorem we have

$$S(1,N,0,1,1,1)=\sum_{n=1}^{N}\frac{(-1)^{n+1}F_{n+1}^2}{F_{n+1}^2(F_{n+1}+2F_n)^2+1}=\frac{1}{2}-\frac{F_{N+1}F_{N+2}}{F_{N+1}^2+F_{N+2}^2}, \tag{2.16}$$

and

$$S(1,\infty,0,1,1,1)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}F_{n+1}^2}{F_{n+1}^2(F_{n+1}+2F_n)^2+1}=\frac{1}{2}\frac{2-\alpha}{2+\alpha}. \tag{2.17}$$

Another example is

$$S(1,N,0,2,1,1)=\sum_{n=1}^{N}\frac{(-1)^{n+1}(F_{n+1}^2+F_{n+2}^2)}{(F_{n+1}^2+F_{n+2}^2+2F_nF_{n+1})^2+1}=\frac{2}{5}-\frac{F_{N+1}F_{N+3}}{F_{N+1}^2+F_{N+3}^2}, \tag{2.18}$$

and

$$S(1,\infty,0,2,1,1)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(F_{n+1}^2+F_{n+2}^2)}{(F_{n+1}^2+F_{n+2}^2+2F_nF_{n+1})^2+1}=\frac{1}{15}. \tag{2.19}$$

3 Application to reciprocal Fibonacci sums

In this section we use the previous theorem to derive some relations between finite and infinite sums containing reciprocal Fibonacci products of order 2. To do so, note that eqs. (2.12) and

(2.13) also hold for $a = 0$. In this case $S(p, N, l, k, r, 0)$ can be evaluated directly by telescoping. However, in view of (2.1)-(2.8) we can write in the finite case

$$\begin{aligned} S(p, N, l, k, r, 0) &= \sum_{n=p}^{N} \frac{\sum_{i=1}^{r} \binom{r}{i} (-1)^{i(n+1+l)} F_k^i F_{n+1+l}^{r-i} F_{n+l+k}^{r-i}}{F_{n+l+k}^r F_{n+1+l+k}^r} \\ &= \frac{(-1)^{p+l+1} F_k F_{N+1-p} \cdot P(F_{p+l} F_{N+1+l+k}, F_{p+l+k} F_{N+1+l})}{F_{p+l+k}^r F_{N+1+l+k}^r}. \end{aligned}$$

Analogously for the infinite case. For $r = 1$ this simplifies to

$$\sum_{n=p}^{N} \frac{(-1)^{n+1}}{F_{n+l+k} F_{n+1+l+k}} = \frac{(-1)^{p+1} F_{N+1-p}}{F_{p+l+k} F_{N+1+l+k}}, \tag{3.1}$$

and

$$\sum_{n=p}^{\infty} \frac{(-1)^{n+1}}{F_{n+l+k} F_{n+1+l+k}} = \frac{(-1)^{p+1}}{\alpha^{p+l+k} F_{p+l+k}}. \tag{3.2}$$

Especially for $p = 1$ and $l = 0$

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{F_{n+k} F_{n+1+k}} = \frac{F_N}{F_{k+1} F_{N+1+k}}, \tag{3.3}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_{n+k} F_{n+1+k}} = \frac{1}{\alpha^{k+1} F_{k+1}}. \tag{3.4}$$

Observe that in subcase $k = 0$ the last two equations coincide with (1.3) and (1.5), respectively. Also, the result for $k = 0$ in (3.4) is established independently in [5].
From $S(1, N, 0, k, 2, 0)$ we obtain

$$\sum_{n=1}^{N} \frac{F_k + 2(-1)^{n+1} F_{n+1} F_{n+k}}{F_{n+k}^2 F_{n+1+k}^2} = \frac{F_N F_{N+1+k} + F_N F_{N+1} F_{k+1}}{F_{k+1}^2 F_{N+1+k}^2}, \tag{3.5}$$

and

$$\sum_{n=1}^{\infty} \frac{F_k + 2(-1)^{n+1} F_{n+1} F_{n+k}}{F_{n+k}^2 F_{n+1+k}^2} = \frac{\alpha^k + F_{k+1}}{\alpha^{2k+1} F_{k+1}^2}, \tag{3.6}$$

from which upon setting $k = 1$ we easily deduce the relationships

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{F_n^2} = \frac{F_{N+1}^2 (F_{N+2}^2 + F_N F_{N+3}) - 1}{2 F_{N+1}^2 F_{N+2}^2} + \frac{(-1)^{N+1}}{F_{N+1}^2} + \frac{(-1)^N}{F_{N+2}^2} - \frac{1}{2} \sum_{n=1}^{N} \frac{1}{F_n^2 F_{n+1}^2}, \tag{3.7}$$

as well as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_n^2} = \frac{1}{2} \Big(\alpha - \sum_{n=1}^{\infty} \frac{1}{F_n^2 F_{n+1}^2} \Big). \tag{3.8}$$

The pair $r=2$ and $k=2$ leads to

$$\begin{aligned}\sum_{n=1}^{N}\frac{(-1)^{n+1}F_n}{F_{n+1}F_{n+2}^2} &= -\frac{3}{8}+\frac{(-1)^{N+1}F_{N+1}}{F_{N+2}F_{N+3}^2}-\frac{F_N(F_{N+3}+2F_{N+1})}{8F_{N+3}^2}\\ &\quad +\frac{F_{N+3}^2+F_{N+1}^2}{2F_{N+1}^2F_{N+2}^2F_{N+3}^2}+\frac{1}{2}\sum_{n=1}^{N}\frac{1}{F_n^2F_{n+1}^2},\end{aligned}\tag{3.9}$$

as well as

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}F_n}{F_{n+1}F_{n+2}^2}=\frac{1}{2}\Big(\sum_{n=1}^{\infty}\frac{1}{F_n^2F_{n+1}^2}-\frac{3\alpha+1}{3\alpha+2}\Big).\tag{3.10}$$

Comparing Eq. (3.10) with Eq. (97) of [5] it is easily verified that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}F_{n+2}}{F_n^2F_{n+1}}=1+\sum_{n=1}^{\infty}\frac{(-1)^nF_n}{F_{n+1}F_{n+2}^2}.\tag{3.11}$$

Likewise, if we take $r=2$ and $k=3$ we obtain

$$\begin{aligned}\sum_{n=1}^{N}\frac{(-1)^{n+1}F_n}{F_{n+2}F_{n+3}^2} &= -\frac{11}{9}+\frac{(-1)^{N+1}F_{N+1}}{F_{N+3}F_{N+4}^2}-\frac{F_N(F_{N+4}+3F_{N+1})}{18F_{N+4}^2}\\ &\quad +\sum_{j=1}^{4}\frac{1}{F_{N+j}^2F_{N+j+1}^2}+\sum_{n=1}^{N}\frac{1}{F_n^2F_{n+1}^2},\end{aligned}\tag{3.12}$$

as well as

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}F_n}{F_{n+2}F_{n+3}^2}=-\frac{2}{\alpha}+\sum_{n=1}^{\infty}\frac{1}{F_n^2F_{n+1}^2}.\tag{3.13}$$

Finally, we consider $S(1,N,0,k,3,0)$. We have

$$F_{N+1+k}^3-F_{N+1}^3F_{k+1}^3=F_kF_N(F_{N+1+k}^2+F_{N+1+k}F_{N+1}F_{k+1}+F_{N+1}^2F_{k+1}^2),$$

and

$$\alpha^{3k}-F_{k+1}^3=\frac{F_k}{\alpha}(\alpha^{2k}+\alpha^kF_{k+1}+F_{k+1}^2).$$

Hence,

$$\sum_{n=1}^{N}\frac{3F_kF_{n+1}F_{n+k}+(-1)^{n+1}(F_k^2+3F_{n+1}^2F_{n+k}^2)}{F_{n+k}^3F_{n+1+k}^3}=\frac{F_N(F_{N+1+k}^2+F_{N+1+k}F_{N+1}F_{k+1}+F_{N+1}^2F_{k+1}^2)}{F_{k+1}^3F_{N+1+k}^3},\tag{3.14}$$

and

$$\sum_{n=1}^{\infty}\frac{3F_kF_{n+1}F_{n+k}+(-1)^{n+1}(F_k^2+3F_{n+1}^2F_{n+k}^2)}{F_{n+k}^3F_{n+1+k}^3}=\frac{\alpha^{2k}+\alpha^kF_{k+1}+F_{k+1}^2}{\alpha^{3k+1}F_{k+1}^3}.\tag{3.15}$$

For $k=1$ we obtain

$$\begin{aligned}\sum_{n=1}^{N}\frac{(-1)^{n+1}(1+3F_n^4)}{F_n^3F_{n+1}^3} &= 1+\frac{(-1)^{N+1}(1+3F_{N+1}^4)}{F_{N+1}^3F_{N+2}^3}+\frac{3}{F_{N+1}F_{N+2}^3}\\ &\quad -\frac{F_N(F_{N+2}F_{N+3}+F_{N+1}^2)}{F_{N+2}^3}+3\sum_{n=1}^{N}\frac{1}{F_nF_{n+1}^3},\end{aligned}\tag{3.16}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1+3F_n^4)}{F_n^3F_{n+1}^3}=2\alpha-3+3\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}^3}. \tag{3.17}$$

For $k=2$ the results are

$$\begin{aligned}\sum_{n=1}^{N}\frac{(-1)^{n+1}(1+3F_n^2F_{n+1}^2)}{F_{n+1}^3F_{n+2}^3} &= \frac{1}{8}+\frac{(-1)^{N+1}(1+3F_{N+1}^2F_{N+2}^2)}{F_{N+2}^3F_{N+3}^3}+\frac{3F_{N+1}}{F_{N+2}^2F_{N+3}^3}\\ &\quad -\frac{F_N(F_{N+3}^2+2F_{N+3}F_{N+1}+4F_{N+1}^2)}{8F_{N+3}^3}+3\sum_{n=1}^{N}\frac{F_n}{F_{n+1}^2F_{n+2}^3},\end{aligned} \tag{3.18}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1+3F_n^2F_{n+1}^2)}{F_{n+1}^3F_{n+2}^3}=\frac{2-\alpha}{(1+\alpha)^2}+3\sum_{n=1}^{\infty}\frac{F_n}{F_{n+1}^2F_{n+2}^3}. \tag{3.19}$$

Finally, inserting $k=3$ we conclude with the identities

$$\begin{aligned}\sum_{n=1}^{N}\frac{(-1)^{n+1}(4+3F_n^2F_{n+2}^2)}{F_{n+2}^3F_{n+3}^3} &= \frac{1}{54}+\frac{(-1)^{N+1}(4+3F_{N+1}^2F_{N+3}^2)}{F_{N+3}^3F_{N+4}^3}+\frac{6F_{N+1}}{F_{N+3}^2F_{N+4}^3}\\ &\quad -\frac{F_N(F_{N+4}^2+3F_{N+1}F_{N+4}+9F_{N+1}^2)}{27F_{N+4}^3}+6\sum_{n=1}^{N}\frac{F_n}{F_{n+2}^2F_{n+3}^3},\end{aligned} \tag{3.20}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(4+3F_n^2F_{n+2}^2)}{F_{n+2}^3F_{n+3}^3}=\frac{1}{2}\frac{2\alpha-3}{8\alpha+5}+6\sum_{n=1}^{\infty}\frac{F_n}{F_{n+2}^2F_{n+3}^3}. \tag{3.21}$$

Disclaimer

Statements and conclusions made in this article are entirely those of the author. They do not necessarily reflect the views of LBBW.

References

[1] André-Jeannin, R. (1990) Lambert Series and the Summation of Reciprocals of Certain Fibonacci-Lucas-Type Sequences, *The Fibonacci Quarterly,* 28(3), 223 - 226.

[2] André-Jeannin, R. (1997) Summation of Reciprocals in Certain Second-Order Recurring Sequences, *The Fibonacci Quarterly,* 35, 68–74.

[3] Brousseau, B.A. (1969) Summation of Infinite Fibonacci Series, *The Fibonacci Quarterly,* 7(2), 143–168.

[4] Carlitz, L. (1971) Reduction Formulas for Fibonacci Summations, *The Fibonacci Quarterly,* 9(5), 449–466, 510–511.

[5] Frontczak, R. (2015) On Infinite Series Involving Fibonacci Numbers, *International Journal of Contemporary Mathematical Sciences,* 10(8), 363–379.

[6] Good, I.J. (1994) A Symmetry Property of Alternating Sums of Products of Reciprocals, *The Fibonacci Quarterly,* 32(3), 284–287.

[7] Grimaldi, R. P. (2012) *Fibonacci and Catalan numbers An Introduction,* John Wiley & Sons, New York.

[8] Horadam, A.F. (1988) Elliptic Functions and Lambert Series in the Summation of Reciprocals in Certain Recurrence-Generated Sequences, *The Fibonacci Quarterly,* 26(2), 98–114.

[9] Melham, R.S. (2002) Reduction Formulas for the Summation of Reciprocals in Certain Second-Order Recurring Sequences, *The Fibonacci Quarterly,* 40(1), 71–75.

[10] Melham, R.S. (2003) On Some Reciprocal Sums of Brousseau: An Alternative Approach to that of Carlitz, *The Fibonacci Quarterly,* 41(1), 59–62.

[11] Rabinowitz, S. (1999) Algorithmic Summation of Reciprocals of Products of Fibonacci Numbers, *The Fibonacci Quarterly,* 37, 122–127.

[12] Udrea, G. (1995) Catalan's Identity and the Chebyshev Polynomials of the Second Kind, *Portugaliae Mathematica,* 52(4), 391–397.