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Series expansions related to the logarithmic mean

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Abstract: We show that the Gregory series combined with Newton's binomial expansion give a natural approach to the logarithmic mean inequalities.
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1 Introduction

In what follows, we let $x \in (-1, 1)$. The well-known series expansion for the logarithmic function

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{1}$$

(discovered for the first time by N. Mercator (1668), see e.g. [3]) applied to "$-x$" in place of x gives

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \tag{2}$$

By substracting equations (1) and (2) we get the Gregory series (J. Gregory (1668), see [3])

$$\log\frac{1+x}{1-x} = 2 \cdot \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \right) \tag{3}$$

The Newton's binomial expansion (stated for the first time by I. Newton (1665), and considered later also by L. Euler (1748) states that for any rational α one has

$$(1+x)^\alpha = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \binom{\alpha}{3}x^3 + \cdots, \tag{4}$$

where $\binom{\alpha}{k} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$ denotes a generalized binomial coefficient.

Particularly, for $\alpha = -\dfrac{1}{2}$ we get $\binom{-1/2}{k} = (-1)^k \cdot \dfrac{1\cdot 3\cdot \ldots \cdot(2k-1)}{2^k \cdot k!}$, and for "$-x^2$" in place of x in (4), we get the expansion

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \cdots + \frac{1\cdot 3\cdot \ldots \cdot (2k-1)}{2^k \cdot k!}x^{2k} + \cdots, \tag{5}$$

which we will need later.

The logarithmic mean of two positive real numbers a and b is defined by

$$L(a,b) = \frac{b-a}{\log b - \log a}(b \neq a); L(a,a) = a \tag{6}$$

This mean has many connections and applications in various domains of Mathematics (see e.g. [1, 2]; [4]–[16]). Particularly, the following classical relations are true for $a \neq b$:

$$G < L < A; \tag{7}$$

$$L < A_{1/3}, \tag{8}$$

where $A_r = A_r(a,b) = \left(\dfrac{a^r + b^r}{2}\right)^{1/r}$; $A = A_1(a,b) = \dfrac{a+b}{2}, G = A_0(a,b) = \lim\limits_{r\to 0} A_r(a,b) = \sqrt{ab}$ are the classical power mean, resp. arithmetic and geometric means of a and b. Many applications to (7) and (8), as well as new proofs are known in the literature. New proofs to (7) and (8), based on integral inequalities have been obtained by the author in [6, 10]. For recent new proofs of (7), see [14, 15]. For the history of (7), see [2, 7, 13].

In what follows, we will show that the Gregory series (3) and Newton's series expansion (5) offer new proofs to (7), as well as to (8). Such a method with series, for the arithmetic–geometric mean of Gauss, appears in [9].

2 The proofs

Since the means L and A_r are homogeneous, it is easy to see that (7), resp. (8) are equivalent to

$$\sqrt{t} < \frac{t-1}{\log t} < \frac{t+1}{2} \tag{7'}$$

respectively,

$$\frac{t-1}{\log t} < \left(\frac{\sqrt[3]{t}+1}{2}\right)^3, \tag{8'}$$

where $t = \dfrac{b}{a} > 1$ (say)

Now, to prove the right side of (7') remark that for $x = \dfrac{t-1}{t+1} \in (0,1)$ in (3) we have, by $t = \dfrac{1+x}{1-x}$ that $\log t = \log \dfrac{1+x}{1-x} > 2x = 2 \cdot \dfrac{t-1}{t+1}$. This gives immediately the right side of (7') for $t > 1$.

For the proof of (8') apply the same idea, by remarking that

$$\log t > 2 \cdot \left(x + \frac{x^3}{3} \right) = 2 \cdot \frac{t-1}{t+1} \cdot \frac{t^2+t+1}{(t+1)^2} \cdot \frac{4}{3}$$

As $(t-1)(t^2+t+1) = t^3 - 1$, we get the inequality

$$\frac{t^3-1}{3\log t} < \left(\frac{t+1}{2} \right)^3 \tag{8''}$$

Now, to get (8') from (8''), it is enough to replace t with $\sqrt[3]{t}$, and the result follows.

For the proof of left side of (7') we will remark first that for the kth terms in the right side of (3) and (5) one has

$$\frac{x^{2k}}{2k+1} < \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{2^k \cdot k!} \cdot x^{2k} \tag{9}$$

Since $0 < x$, it is sufficient to prove the inequality

$$1 \cdot 3 \cdot \ldots \cdot (2k-1)(2k+1) > 2^k \cdot k! \tag{10}$$

This follows immediately e.g. by mathematical induction. For $k = 1, 2$ it is true; and assuming it for k, we get

$$1 \cdot 3 \cdot \ldots \cdot (2k+1)(2k+3) > 2^k \cdot k!(2k+3) > 2^{k+1}(k+1)!,$$

where the last inequality holds by $2k+3 > 2k+2$.

Therefore, we get the inequality

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots < x \left(1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \cdots \right), \tag{11}$$

and by letting $\dfrac{1+x}{1-x} = t > 1$ we obtain

$$\log t < 2 \cdot \left(\frac{t-1}{t+1} \right) \cdot \frac{t+1}{2\sqrt{t}},$$

so the left side of (7') follows, too.

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