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On function "Restrictive factor"

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Abstract: Some new properties of the arithmetic function called "Restrictive factor" are formulated and studied.
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1 Introduction

In 2002, the author introduced an arithmetic function, called "Restrictive factor" and studied some of its properties in [1]. It was an object of research of L. Panaitopol [2], P. Spiegelhalter and A. Zaharescu [4]. Here, we study some new properties of the Restrictive factor. We use the following notation, following [3].

- $\mathcal{N}$ is the set of all positive integers. If $n > 1$ and $n \in \mathcal{N}$, then n has the form
$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$
that is called a cannonical factorization of n, where $k, \alpha_1, ..., \alpha_k \in \mathcal{N}, k \geq 1, p_1, ..., p_k$ are different primes; in some cases, it is suitable the order $p_1 < ... < p_k$ to be valid;

- $\underline{set}(n) = \{p_1, p_2, ..., p_k\}$,

- $\omega(n) = k$ is the number of the distinct prime divisors of $n \in \mathcal{N}, n > 1$ and $\omega(1) = 0$;

- $\Omega(n) = \sum_{i=1}^{k} \alpha_i$,
- $\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1}-1}{p_i-1}$ is the sum of all different divisors of n and $\sigma(1) = 1$,
- $\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i - 1),\ \varphi(1) = 1$ is Euler's totient function;
- ψ is Dedekind's function, which is multiplicative and
$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i + 1)$$
for $n > 1, n \in \mathcal{N}$ and $\psi(1) = 1$.

2 Main results

In [1], we juxtaposed to natural number n the (natural) number

$$RF(n) = \prod_{i=1}^{k} p_i^{\alpha_i-1}$$

that we called *Restrictive Factor.*

Let us define
$$RF(1) = 0.$$

It can be easily seen that if for every i $(1 \leq i \leq k)$ $\alpha_i = 1$, then $RF(n) = 1$.

On the other hand, if there is at least one $\alpha_i > 1$, then

$$n > RF(n) > 1.$$

In [1], it is proved that for every natural number n:

$$n^2 - \varphi(n).\sigma(n) \geq RF(n)$$

and
$$\varphi(n) + \sigma(n) - 2n \geq RF(n).$$

First, for each natural number n, we see that

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i - 1) > \prod_{i=1}^{k} p_i^{\alpha_i-1} = RF(n)$$

and
$$n - \varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i-1} \left(\prod_{i=1}^{k} p_i - \prod_{i=1}^{k} (p_i - 1) \right) > \prod_{i=1}^{k} p_i^{\alpha_i-1} = RF(n).$$

From the well-know inequality

$$\sigma(n) + \varphi(n) \geq 2n,$$

it follows that

$$\sigma(n) - n \geq RF(n).$$

It is seen easily that there natural numbers n for which $n - \varphi(n) > \varphi(n)$ and others for which the opposite inequality is valid. For example,

$$15 - \varphi(15) = 7 < 8 = f(15)$$

and

$$30 - f(30) = 22 > 8 = f(30).$$

Therefore,

$$RF(n) \leq \min(n - \varphi(n), \varphi(n)).$$

Second, let

$$\min(n) = p_1,$$

$$\max(n) = p_k.$$

Then,

$$RF(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1} \geq \prod_{i=1}^{k} \min(n)^{\alpha_i - 1} = \min(n)^{\left(\sum\limits_{i=1}^{k} \alpha_i\right) - k}$$
$$= \min(n)^{\Omega(n) - \omega(n)}$$

and

$$RF(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1} \leq \prod_{i=1}^{k} \max(n)^{\alpha_i - 1} = \max(n)^{\left(\sum\limits_{i=1}^{k} \alpha_i\right) - k}$$
$$= \max(n)^{\Omega(n) - \omega(n)}.$$

Therefore,

$$\min(n)^{\Omega(n) - \omega(n)} \leq RF(n) \leq \max(n)^{\Omega(n) - \omega(n)}.$$

Third, let

$$n = \left(\prod_{i=1}^{k} p_i^{\alpha_i}\right) \cdot \left(\prod_{i=k+1}^{k+l} p_i\right),$$

where $p_1, ..., p_{k+1}$ are different prime numbers and $\alpha_1, ..., \alpha_k \geq 2$. Then, the following assertions are valid.

Theorem 1. For each natural number n:

(a) $\frac{\varphi(RF(n))}{RF(n)} > \frac{\varphi(n)}{n}$,

(b) $\frac{\psi(n)}{n} > \frac{\psi(RF(n))}{RF(n)}$,

Proof. (a) Let the natural number n be given. Then

$$\frac{\varphi(RF(n))}{RF(n)} - \frac{\varphi(n)}{n} = \frac{\varphi(\prod_{i=1}^{k} p_i^{\alpha_i - 1})}{\prod_{i=1}^{k} p_i^{\alpha_i - 1}} - \frac{\varphi\left(\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)\cdot\left(\prod_{i=k+1}^{k+l} p_i\right)\right)}{\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)\cdot\left(\prod_{i=k+1}^{k+l} p_i\right)}$$

$$= \frac{\prod_{i=1}^{k} p_i^{\alpha_i - 2}(p_i - 1)}{\prod_{i=1}^{k} p_i^{\alpha_i - 1}} - \frac{\left(\prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1)\right)\cdot\left(\prod_{i=k+1}^{k+l} (p_i - 1)\right)}{\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)\cdot\left(\prod_{i=k+1}^{k+l} p_i\right)}$$

$$= \frac{\prod_{i=1}^{k} (p_i - 1)}{\prod_{i=1}^{k} p_i} - \frac{\prod_{i=1}^{k+l} (p_i - 1)}{\prod_{i=1}^{k+l} p_i}$$

$$= \frac{\prod_{i=1}^{k} (p_i - 1)}{\prod_{i=1}^{k} p_i}\left(1 - \frac{\prod_{i=k+1}^{k+l} (p_i - 1)}{\prod_{i=k+1}^{k+l} p_i}\right) > 0.$$

(b) is proved by analogy. □

Theorem 2. For each natural number n:

$$\frac{\sigma(n)}{n} > \frac{\sigma(RF(n))}{RF(n)}.$$

Proof. Let the natural number n be given. Then

$$\frac{\sigma(n)}{n} - \frac{\sigma(RF(n))}{RF(n)} = \frac{\sigma\left(\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)\cdot\left(\prod_{i=k+1}^{k+l} p_i\right)\right)}{\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)\cdot\left(\prod_{i=k+1}^{k+l} p_i\right)} - \frac{\sigma(\prod_{i=1}^{k} p_i^{\alpha_i - 1})}{\prod_{i=1}^{k} p_i^{\alpha_i - 1}}$$

$$= \frac{\left(\prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}\right)\cdot\left(\prod_{i=k+1}^{k+l} (p_i + 1)\right)}{\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)\cdot\left(\prod_{i=k+1}^{k+l} p_i\right)} - \frac{\prod_{i=1}^{k} \frac{p_i^{\alpha_i} - 1}{p_i - 1}}{\prod_{i=1}^{k} p_i^{\alpha_i - 1}}$$

(from $\frac{q^{b+1}-1}{q-1} > q\frac{q^b-1}{q-1}$ for every natural numbers $q \geq 2$ and $b \geq 1$)

$$> \frac{\left(\prod_{i=1}^{k} p_i \frac{p_i^{\alpha_i} - 1}{p_i - 1}\right)\cdot\left(\prod_{i=k+1}^{k+l} (p_i + 1)\right)}{\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)\cdot\left(\prod_{i=k+1}^{k+l} p_i\right)} - \frac{\prod_{i=1}^{k} \frac{p_i^{\alpha_i} - 1}{p_i - 1}}{\prod_{i=1}^{k} p_i^{\alpha_i - 1}}$$

$$\frac{\prod_{i=1}^{k} \frac{p_i^{\alpha_i} - 1}{p_i - 1}}{\prod_{i=1}^{k} p_i^{\alpha_i - 1}}\left(\frac{\prod_{i=k+1}^{k+l} (p_i + 1)}{\prod_{i=k+1}^{k+l} p_i} - 1\right) > 0.$$ □

Fourth, we prove

Theorem 3. For each natural number n:

$$\sigma(n) - \psi(n) \geq RF(RF(n)) \tag{1}$$

Proof. Let n be a prime number. Then

$$\sigma(n) - \psi(n) - RF(RF(n)) = (n+1) - (n+1) - RF(1) = 0 - 0 = 0.$$

Let us assume that (1) is valid for some natural number n for which $\Omega(n) = s$ and let p be a prime number. For p there are two cases.

Case 1: $p \notin \underline{set}(n)$. Then

$$\sigma(np) - \psi(np) - RF(RF(np)) = \sigma(n)(p+1) - \psi(n)(p+1) - RF(RF(n))$$
$$> (p+1)(\sigma(n) - \psi(n) - RF(RF(n))) \geq 0.$$

Case 2: $p \in \underline{set}(n)$. Then $n = mp^a$ for some natural numbers $a > 1$ and m with $\Omega(m) < s$, and

$$\sigma(np) - \psi(np) - RF(RF(np)) = \sigma(m)\frac{p_i^{\alpha_i+2} - 1}{p_i - 1} - \psi(m)p^{\alpha_i}(p_i+1) - RF(RF(m)p^{\alpha_i})$$
$$= \sigma(m)(p_i^{\alpha_i+1} + p_i^{\alpha_i} + ... + 1) - \psi(m)(p^{\alpha_i+1} + p_i^{\alpha_i}) - RF(RF(m))p^{\alpha_i-1} > 0,$$

because by induction (1) is valid. □

Theorem 4. For each natural number n:

$$RF(n) \geq \frac{\sigma(n)}{\psi(n)}. \tag{2}$$

Proof. Let n be a prime number. Then

$$RF(n) - \frac{\sigma(n)}{\psi(n)} = 1 - \frac{n+1}{n+1} = 0.$$

Let us assume that (2) is valid for some natural number n for which $\Omega(n) = s$ and let p be a prime number. For p there are two cases.

Case 1: $p \notin \underline{set}(n)$. Then

$$RF(np) - \frac{\sigma(np)}{\psi(np)} = RF(n) - \frac{\sigma(n)(p+1)}{\psi(n)(p+1)} = RF(n) - \frac{\sigma(n)}{\psi(n)} \geq 0.$$

Case 2: $p \in \underline{set}(n)$. Then $n = mp^a$ for some natural numbers $a > 1$ and m with $\Omega(m) < s$, and

$$RF(np) - \frac{\sigma(np)}{\psi(np)} = RF(m)p^a - \frac{\sigma(m)\frac{p^{a+2}-1}{p-1}}{\psi(m)p^a(p+1)}$$
$$= RF(m)p^a - \frac{\sigma(m)(p^{a+2}-1)}{\psi(m)p^a(p^2-1)} \geq p^a\left(RF(m) \geq \frac{\sigma(m)}{\psi(m)}\right) > 0,$$

because by induction (2) is valid. □

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