Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 22, 2016, No. 1, 55–58

Extended Pascal's triangle

Krassimir T. Atanassov

Department of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 105, Sofia-1113, Bulgaria

and

Intelligent Systems Laboratory
Prof. Asen Zlatarov University, Bourgas-8000, Bulgaria

e-mail: krat@bas.bg

In memory of Assoc. Prof. Rusi Rusev (1934–2016)

Received: 20 February 2015 **Revised:** 4 October 2015 **Accepted:** 14 October 2015

Abstract: An extension of the best known Pascal's triangle is introduced and an explicit formula for its members is given.
Keywords: Binomial coefficients, Pascal's triangle.
AMS Classification: 11B65.

1 Introduction

The Pascal's triangle

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & & \vdots & & & \vdots & & & \vdots & & \end{array}$$

is one of the best known mathematical objects. In a series of papers, collected in my book [1], some of its modifications are described. Here, an extendsion of the Pascal's triangle is introduced and an explicit formula for its elements is given.

2 Main result

Let us have a constant a and two sequences $\{b_i\}_{i\geq 1}$ and $\{c_i\}_{i\geq 1}$ that comprise of real numbers. Then, we can construct the following triangle

$$\begin{array}{ccccccccc}
 & & & & a & & & & \\
 & & & b_1 & & c_1 & & & \\
 & & b_2 & & b_1+c_1 & & c_2 & & \\
 & b_3 & & b_1+b_2+c_1 & & b_1+c_1+c_2 & & c_3 & \\
b_4 & & b_1+b_2+b_3+c_1 & & 2b_1+b_2+2c_1+c_2 & & b_1+c_1+c_2+c_3 & & c_4 \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

which we call *"extended Pascal's triangle"*.

Let us denote its elements by $\alpha_{m,n}$.

For the needs of the statement below, we must define the following sequences:

$$\begin{array}{lccccc}
\{\varphi_{0,j}\}_{j\geq 1}: & 1, & 0, & 0, & 0, & ... \\
\{\varphi_{1,j}\}_{j\geq 1}: & 1, & 1, & 1, & 1, & ... \\
\{\varphi_{2,j}\}_{j\geq 1}: & 1, & 2, & 3, & 4, & ... \\
\{\varphi_{3,j}\}_{j\geq 1}: & 1, & 3, & 6, & 10, & ... \\
\{\varphi_{4,j}\}_{j\geq 1}: & 1, & 4, & 10, & 20, & ...
\end{array}$$

such that for every natural number $i \geq 0$:

$$\varphi_{i+1,j} = \sum_{k=1}^{j} \varphi_{i,k}. \tag{1}$$

It is seen easily that

$$\begin{array}{lccccc}
\{\varphi_{1,j}\}_{j\geq 1}: & C_0^0, & C_1^1, & C_2^2, & C_3^3, & ... \\
\{\varphi_{2,j}\}_{j\geq 1}: & C_1^0, & C_2^1, & C_3^2, & C_4^3, & ... \\
\{\varphi_{3,j}\}_{j\geq 1}: & C_2^0, & C_3^1, & C_4^2, & C_5^3, & ... \\
\{\varphi_{4,j}\}_{j\geq 1}: & C_3^0, & C_4^1, & C_5^2, & C_6^3, & ...
\end{array},$$

where

$$C_n^k = \binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}.$$

Therefore, for the natural numbers $i, j \geq 1$:

$$\varphi_{i,j} = C_{i+j-2}^{j-1}. \tag{2}$$

Below, we keep the φ-notation, because these sequences have some interesting properties that we will discuss in a next paper.

Theorem. For every two natural numbers $m \geq 1$ and $0 \leq n \leq m$:

$$\alpha_{m,n} = \sum_{i=0}^{m-n-1} \varphi_{n,m-n-i} b_{i+1} + \sum_{i=1}^{n} \varphi_{m-n,n-i+1} c_i. \tag{3}$$

Proof. The α-elements satisfy the well-known property of the members of the Pascal's triangle

$$\alpha_{m,n-1} + \alpha_{m,n} = \alpha_{m+1,n}. \tag{4}$$

So, here we check this property.

First, we see directly that

$$\alpha_{1,0} = \sum_{i=0}^{0} \varphi_{0,1-i} b_{i+1} + \sum_{i=1}^{0} \varphi_{0,1-i} c_i = b_1,$$

$$\alpha_{1,1} = \sum_{i=0}^{-1} \varphi_{1,-i} b_{i+1} + \sum_{i=1}^{1} \varphi_{0,2-i} c_i = c_1.$$

Let us assume that all elements of the m-th row of the extended Pascal's triangle satisfy (3) for $m \geq 2$. Now, using (4), we calculate for an arbitrary natural number n $(1 \leq n \leq m-1)$ that

$$\alpha_{m+1,n} = \alpha_{m,n-1} + \alpha_{m,n}$$

$$= \sum_{i=0}^{m-n} \varphi_{n-1,m-n-i+1} b_{i+1} + \sum_{i=1}^{n-1} \varphi_{m-n+1,n-i} c_i + \sum_{i=0}^{m-n-1} \varphi_{n,m-n-i} b_{i+1} + \sum_{i=1}^{n} \varphi_{m-n,n-i+1} c_i$$

$$= (\varphi_{n-1,m-n+1} + \varphi_{n,m-n}) b_1 + (\varphi_{n-1,m-n} + \varphi_{n,m-n-1}) b_2 + ... + (\varphi_{n-1,2} + \varphi_{n,1}) b_{m-n} + \varphi_{n-1,1} b_{m-n+1}$$

$$+ (\varphi_{m-n+1,n-1} + \varphi_{m-n,n}) c_1 + (\varphi_{m-n+1,n-2} + \varphi_{m-n,n-1}) c_2 + ... + (\varphi_{m-n+1,1} + \varphi_{m-n,2}) c_{n-1} + \varphi_{m-n,1} c_n$$

(from (1))

$$= (\sum_{k=1}^{m-n} \varphi_{n-1,k} + \varphi_{n-1,m-n+1}) b_1 + ... + (\sum_{k=1}^{1} \varphi_{n-1,k} + \varphi_{n-1,2}) b_{m-n} + \varphi_{n-1,1} b_{m-n+1}$$

$$+ (\sum_{k=1}^{n-1} \varphi_{m-n,k} + \varphi_{m-n,n}) c_1 + ... + (\sum_{k=1}^{1} \varphi_{m-n,k} + \varphi_{m-n,2}) c_{n-1} + \varphi_{m-n,1} c_n$$

$$= (\sum_{k=1}^{m-n+1} \varphi_{n-1,k}) b_1 + ... + (\sum_{k=1}^{2} \varphi_{n-1,k}) b_{m-n} + \varphi_{n-1,1} b_{m-n+1}$$

$$+ (\sum_{k=1}^{n} \varphi_{m-n,k}) c_1 + ... + (\sum_{k=1}^{2} \varphi_{m-n,k}) c_{n-1} + \varphi_{m-n,1} c_n$$

(from (1) and from $\varphi_{n-1,1} = \varphi_{n,1}$ for each natural number n)

$$= \varphi_{n,m-n+1}b_1 + ... + \varphi_{n,2}b_{m-n} + \varphi_{n,1}b_{m-n+1}$$

$$+\varphi_{m-n+1,n}c_1 + ... + \varphi_{m-n+1,2}c_{n-1} + \varphi_{m-n+1,1}c_n$$

$$= \sum_{i=0}^{m-n} \varphi_{n,m-n-i+1}b_{i+1} + \sum_{i=1}^{n} \varphi_{m-n+1,n-i+1}c_i$$

This completes the proof of (3). □

Using (2), we can see that (3) has the form:

$$\alpha_{m,n} = \sum_{i=0}^{m-n-1} C_{m-i-2}^{m-n-i-1} b_{i+1} + \sum_{i=1}^{n} C_{m-i-1}^{m-n} c_i.$$

References

[1] Atanassov, K. (2015) *On Some Pascals Like Triangles,* Warsaw School of Information Technology of the Polish Academy of Sciences, Warsaw.