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Inverse of triangular matrices and generalized bivariate Fibonacci and Lucas p-polynomials

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Abstract: In this paper, we compute the generalized bivariate Fibonacci and Lucas p-polynomials by using inverse of various triangular matrices. In addition, in each calculation, instead of obtaining a type of sequence only, we are able to determine successive n terms of the two types of polynomial sequences simultaneously.

Keywords: Generalized bivariate Fibonacci and Lucas p-polynomials, Triangular matrix.
AMS Classification: Primary: 11B37; Secondary: 15A15, 15A51.

1 Introduction

The generalized bivariate Fibonacci and Lucas p-polynomials [18] are defined by

$$F_{p,n}(x,y) = xF_{p,n-1}(x,y) + yF_{p,n-p-1}(x,y) \tag{1.1}$$

for $n > p$, with boundary conditions $F_{p,0}(x,y) = 0$, $F_{p,n}(x,y) = x^{n-1}$, $n = 1,2,...,p$ and

$$L_{p,n}(x,y) = xL_{p,n-1}(x,y) + yL_{p,n-p-1}(x,y)$$

for $n > p$, with boundary conditions $L_{p,0}(x,y) = (p+1)$, $L_{p,n}(x,y) = x^n$, $n = 1,2,...,p$.

It is not difficult to see that generalized bivariate Fibonacci and Lucas p-polynomials are general forms of the Fibonacci, Lucas, Pell, Jacobsthal, Pell–Lucas, Jacobsthal–Lucas sequences as well as Fibonacci, Lucas, Pell, Jacobsthal, Pell–Lucas, Jacobsthal–Lucas, bivariate Fibonacci and Lucas, first and second types of Chebyshev polynomials and many others.

These sequences and polynomials have a grate number of application areas in applied mathematics and physics. The bivariate Fibonacci and Lucas p-polynomials are widely used in theoretical physics for modeling physical processes [18]. Since, obtaining the desired terms of these sequences and polynomials are very important, numerous researchers studied on them, see [2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22].

Chen and Yu [1] presented some matrices and obtained some relations between them as follows:

$$H = \begin{bmatrix} h_{11} & h_{12} & 0 & \cdots & 0 \\ h_{21} & h_{22} & h_{23} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ h_{n-1,1} & h_{n-1,2} & \cdots & h_{n-1,n-1} & h_{n-1,n} \\ h_{n,1} & h_{n,2} & \cdots & h_{n,n-1} & h_{n,n} \end{bmatrix},$$

$$\widetilde{H} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ h_{11} & h_{12} & 0 & \ddots & \vdots & 0 \\ h_{21} & h_{22} & h_{23} & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ h_{n-1,1} & h_{n-1,2} & \cdots & h_{n-1,n-1} & h_{n-1,n} & 0 \\ h_{n,1} & h_{n,2} & \cdots & h_{n,n-1} & h_{n,n} & 1 \end{bmatrix},$$

$$\widetilde{H}^{-1} = \left[\begin{array}{c|c} [\alpha]_{n\times 1} & [L]_{n\times n} \\ \hline h & [\beta^T]_{1\times n} \end{array} \right]_{(n+1)\times(n+1)} \tag{1.2}$$

$$\det(H) = (-1)^n h . \det(\widetilde{H}), \tag{1.3}$$

$$L = H^{-1} + h^{-1}\alpha\beta^T, \tag{1.4}$$

$$H\alpha + he_n = 0 \tag{1.5}$$

where e_n is the n-th column of the identity matrix I_n.

Theorem 1.1. *[7]Let e_1 be the first column of the identity matrix I_n and L, β, h be the matrices described in the equality* (1.2). *Then,*

$$\beta^T H + he_1 = 0. \tag{1.6}$$

Now we give another proof of Theorem 2.5. in [4]. Note that the same procedure can be applied to any linear recurrence relation of order n.

Theorem 1.2. *[4]Let $p \geq 1$ be an integer, $F_{p,n}(x,y)$ be the generalized bivariate Fibonacci p-polynomials and $N_{p,n} = (n_{ij})$ an $n \times n$ Hessenberg matrix defined by*

$$n_{ij} = \begin{cases} -1 & \textit{if } j = i+1, \\ x & \textit{if } i = j, \\ y & \textit{if } i - j = p, \\ 0 & \textit{otherwise.} \end{cases} \tag{1.7}$$

Then,

$$\det(N_{p,n}) = F_{p,n+1}(x,y). \tag{1.8}$$

Proof. We proceed by induction on n. The result clearly holds for $n = 1$. Now suppose that the result is true for all positive integers less than or equal to $n-1$. We prove it for n. From Equation (1.1), we have $N_{p,n}[F_{p,1}(x,y), F_{p,2}(x,y), \ldots, F_{p,n}(x,y)]^T = [0, 0, \ldots, 0, F_{p,n+1}(x,y)]^T$ and by Cramer's rule we get

$$F_{p,n}(x,y) = \frac{\det(N_{p,n-1})F_{p,n+1}(x,y)}{\det(N_{p,n})}.$$

So

$$F_{p,n+1}(x,y) = \frac{\det(N_{p,n})F_{p,n}(x,y)}{\det(N_{p,n-1})}$$

and from the hypothesis of induction we obtain $\det(N_{p,n}) = F_{p,n+1}(x,y)$. $\square$

Theorem 1.3. *[4]Let $F_{p,n}(x,y)$ be the generalized bivariate Fibonacci p-polynomials and $R_{p,n} = (r_{st})$ an $n \times n$ Hessenberg matrix defined by*

$$r_{st} = \begin{cases} i & \textit{if } s = t-1, \\ x & \textit{if } s = t, \\ i^p y & \textit{if } p = s - t, \\ 0 & \textit{otherwise.} \end{cases} \tag{1.9}$$

Then,

$$\det(R_{p,n}) = F_{p,n+1}(x,y) \tag{1.10}$$

where $n \geq 1$ and $i = \sqrt{-1}$.

Theorem 1.4. *[5]Let $L_{p,n}(x,y)$ be the generalized bivariate Lucas p-polynomials and $W_{p,n} = (w_{st})$ be an $n \times n$ Hessenberg matrix defined by*

$$w_{st} = \begin{cases} i & \textit{if } s = t-1, \\ x & \textit{if } s = t, \\ i^p y & \textit{if } p = s - t \textit{ and } t \neq 1, \\ (p+1)i^p y & \textit{if } p = s - t \textit{ and } t = 1, \\ 0 & \textit{otherwise.} \end{cases} \tag{1.11}$$

Then,

$$\det(W_{p,n}) = L_{p,n}(x,y) \tag{1.12}$$

where $n \geq 1$ and $i = \sqrt{-1}$.

Theorem 1.5. *[5]Let $p \geq 1$ be an integer, $L_{p,n}(x,y)$ be the generalized bivariate Lucas p-polynomials and $M_{p,n} = (m_{ij})$ be an $n \times n$ Hessenberg matrix defined by*

$$m_{ij} = \begin{cases} -1 & \text{if } j = i+1, \\ x & \text{if } i = j, \\ y & \text{if } p = i - j \text{ and } j \neq 1, \\ (p+1)y & \text{if } p = i - j \text{ and } j = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.13}$$

Then,

$$\det(M_{p,n}) = L_{p,n}(x,y). \tag{1.14}$$

The following part includes matrix representations of generalized bivariate Fibonacci and Lucas p-polynomials, which continues the earlier papers [7, 19]. The advantage of the calculations in this study is getting larger number of terms of the sequences at a time, i.e., taking an inverse of a specially prepared triangular matrice gives us successive n terms of both the generalized bivariate Fibonacci and Lucas p-polynomials simultaneously.

2 Main results

Theorem 2.1. *Let $p \geq 1$ be an integer, $R_{p,n}$ be an $n \times n$ Hessenberg matrix in* (1.9) *and*

$$\widetilde{R}_{p,n} = \begin{bmatrix} 1 & 0 & 0 \\ & & \\ & R_{p,n} & 0 \\ & & 1 \end{bmatrix}.$$

Then, $(\widetilde{R}_{p,n})^{-1} =$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ iF_{p,2}(x,y) & -i & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ i^{(n-3)}F_{p,n-2}(x,y) & i^{(n-1)}F_{p,n-3}(x,y) & \ddots & \ddots & 0 & 0 \\ i^{(n-2)}F_{p,n-1}(x,y) & i^{(n)}F_{p,n-2}(x,y) & i^{(n-4)}F_{p,n-3}(x,y) & \ddots & 0 & 0 \\ i^{(n-1)}F_{p,n}(x,y) & i^{(n+1)}F_{p,n-1}(x,y) & i^{(n-3)}F_{p,n-2}(x,y) & \ddots & -i & 0 \\ i^{(n+1)}F_{p,n+1}(x,y) & i^{(n-1)}F_{p,n}(x,y) & i^{(n-2)}F_{p,n-1}(x,y) & \cdots & iF_{p,2}(x,y) & 1 \end{bmatrix}$$

where $i = \sqrt{-1}$.

Proof. Firstly, construct the matrix

$$(\widetilde{R}_{p,n})^{-1} = \left[\begin{array}{c|c} [\alpha]_{n\times 1} & [L]_{n\times n} \\ \hline h & \left[\beta^T\right]_{1\times n} \end{array}\right]_{(n+1)\times(n+1)}$$

and obtain the entries.
$i)$ We obtain h by using (1.3) and (1.10):

$$\begin{aligned}\det(R_{p,n}) &= (-1)^n h.\det(\widetilde{R}_{p,n}) \\ \Rightarrow h &= \frac{\det(R_{p,n})}{(-1)^n\det(\widetilde{R}_{p,n})} = \frac{F_{p,n+1}(x,y)}{(-1)^n(i)^{n-1}} = (i)^{n+1}F_{p,n+1}(x,y).\end{aligned}$$

$ii)$ We obtain matrices $[\alpha]$ and $[\beta]$ by using (1.5) and (1.6):

$$\begin{aligned}[\alpha] &= -(R_{p,n})^{-1}(i)^{n+1}F_{p,n+1}(x,y)e_n \\ &= \begin{bmatrix} F_{p,1}(x,y) \\ iF_{p,2}(x,y) \\ \vdots \\ i^{(n-2)}F_{p,n-1}(x,y) \\ i^{(n-1)}F_{p,n}(x,y) \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}[\beta^T] &= -(i)^{n+1}F_{p,n+1}(x,y)e_1(R_{p,n})^{-1} \\ \Rightarrow & \ [\beta^T] = \begin{bmatrix} i^{(n-1)}F_{p,n}(x,y) & i^{(n-2)}F_{p,n-1}(x,y) & \cdots & iF_{p,2}(x,y) & F_{p,1}(x,y) \end{bmatrix}.\end{aligned}$$

$iii)$ We obtain matrix $[L]$ by using (1.4):

$$\begin{aligned}[L] &= (R_{p,n})^{-1} + ((i)^{n+1}F_{p,n+1}(x,y))^{-1}\alpha\beta^T \\ &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -i & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ i^{(n-1)}F_{p,n-3}(x,y) & \ddots & \ddots & 0 & 0 \\ i^{(n)}F_{p,n-2}(x,y) & i^{(n-4)}F_{p,n-3}(x,y) & \ddots & 0 & 0 \\ i^{(n+1)}F_{p,n-1}(x,y) & i^{(n-3)}F_{p,n-2}(x,y) & \ddots & -i & 0 \end{bmatrix}\end{aligned}$$

Consequently, we obtain the required result. □

Example 2.2. *We obtain some terms of generalized bivariate Fibonacci-4 polynomials, for $n = 6$ by using Theorem 2.1.*

$$\widetilde{R}_{4,6} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & i & 0 & 0 & 0 & 0 & 0 \\ 0 & x & i & 0 & 0 & 0 & 0 \\ 0 & 0 & x & i & 0 & 0 & 0 \\ 0 & 0 & 0 & x & i & 0 & 0 \\ i^4y & 0 & 0 & 0 & x & i & 0 \\ 0 & i^4y & 0 & 0 & 0 & x & 1 \end{bmatrix}$$

and inverse of $\widetilde{R}_{4,6}$ is

$$(\widetilde{R}_{4,6})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ ix & -i & 0 & 0 & 0 & 0 & 0 \\ -x^2 & x & -i & 0 & 0 & 0 & 0 \\ -ix^3 & ix^2 & x & -i & 0 & 0 & 0 \\ x^4 & -x^3 & ix^2 & x & -i & 0 & 0 \\ iy+ix^5 & -ix^4 & -x^3 & ix^2 & x & -i & 0 \\ -2ixy-ix^6 & iy+ix^5 & x^4 & -ix^3 & -x^2 & ix & 1 \end{bmatrix}$$

$$= \begin{bmatrix} F_{4,1}(x,y) & 0 & 0 & 0 & 0 & 0 & 0 \\ iF_{4,2}(x,y) & -iF_{4,1}(x,y) & 0 & 0 & 0 & 0 & 0 \\ -F_{4,3}(x,y) & F_{4,2}(x,y) & -iF_{4,1}(x,y) & 0 & 0 & 0 & 0 \\ -iF_{4,4}(x,y) & iF_{4,3}(x,y) & F_{4,2}(x,y) & -iF_{4,1}(x,y) & 0 & 0 & 0 \\ F_{4,5}(x,y) & -F_{4,4}(x,y) & iF_{4,3}(x,y) & F_{4,2}(x,y) & -iF_{4,1}(x,y) & 0 & 0 \\ iF_{4,6}(x,y) & -iF_{4,5}(x,y) & -F_{4,4}(x,y) & iF_{4,3}(x,y) & F_{4,2}(x,y) & -iF_{4,1}(x,y) & 0 \\ -iF_{4,7}(x,y) & iF_{4,6}(x,y) & F_{4,5}(x,y) & -iF_{4,4}(x,y) & -F_{4,3}(x,y) & iF_{4,2}(x,y) & 1 \end{bmatrix}.$$

Theorem 2.3. *Let $p \geq 1$ be an integer, $N_{p,n}$ be an $n \times n$ Hessenberg matrix in* (1.7) *and*

$$\widetilde{N}_{p,n} = \begin{bmatrix} 1 & 0 & 0 \\ & & \\ & -N_{p,n} & 0 \\ & & 1 \end{bmatrix}.$$

Then, $(\widetilde{N}_{p,n})^{-1} =$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ F_{p,2}(x,y) & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ F_{p,n-2}(x,y) & F_{p,n-3}(x,y) & \ddots & \ddots & 0 & 0 \\ F_{p,n-1}(x,y) & F_{p,n-2}(x,y) & F_{p,n-3}(x,y) & \ddots & 0 & 0 \\ F_{p,n}(x,y) & F_{p,n-1}(x,y) & F_{p,n-2}(x,y) & \ddots & 1 & 0 \\ F_{p,n+1}(x,y) & F_{p,n}(x,y) & F_{p,n-1}(x,y) & \cdots & F_{p,2}(x,y) & 1 \end{bmatrix}.$$

Proof. Proof is similar to the proof of Theorem 2.1, by using (1.8). □

Now we obtain two matrices of which first columns include generalized bivariate Lucas p-polynomials and last rows include generalized bivariate Fibonacci p-polynomials. That is, we obtain n successive terms of these two sequences simultaneously.

Theorem 2.4. *Let $p \geq 1$ be an integer, $W_{p,n}$ be an $n \times n$ Hessenberg matrix in* (1.11) *and*

$$\widetilde{W}_{p,n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & & & \vdots \\ & W_{p,n} & & 0 \\ & & & 1 \end{bmatrix}.$$

Then, $(\widetilde{W}_{p,n})^{-1} =$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ iL_{p,1}(x,y) & -i & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ i^{(n-3)}L_{p,n-3}(x,y) & i^{(n-1)}F_{p,n-3}(x,y) & \ddots & \ddots & 0 & 0 \\ i^{(n-2)}L_{p,n-2}(x,y) & i^{(n)}F_{p,n-2}(x,y) & i^{(n-4)}F_{p,n-3}(x,y) & \ddots & 0 & 0 \\ i^{(n-1)}L_{p,n-1}(x,y) & i^{(n+1)}F_{p,n-1}(x,y) & i^{(n-3)}F_{p,n-2}(x,y) & \ddots & -i & 0 \\ i^{(n+1)}L_{p,n}(x,y) & i^{(n-1)}F_{p,n}(x,y) & i^{(n-2)}F_{p,n-1}(x,y) & \cdots & iF_{p,2}(x,y) & 1 \end{bmatrix}$$

where $i = \sqrt{-1}$.

Proof. Proof is similar to the proof of Theorem 2.1, by using (1.12). $\square$

Example 2.5. *We obtain some terms of generalized bivariate Fibonacci-2 and Lucas-2 polynomials, by using Theorem 2.4.*

$$\widetilde{W}_{2,5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ x & i & 0 & 0 & 0 & 0 \\ 0 & x & i & 0 & 0 & 0 \\ 3i^2y & 0 & x & i & 0 & 0 \\ 0 & i^2y & 0 & x & i & 0 \\ 0 & 0 & i^2y & 0 & x & 1 \end{bmatrix}$$

and inverse of $\widetilde{W}_{2,5}$ *is*

$$(\widetilde{W}_{2,5})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ ix & -i & 0 & 0 & 0 & 0 \\ -x^2 & x & -i & 0 & 0 & 0 \\ -3iy - ix^3 & ix^2 & x & -i & 0 & 0 \\ 4xy + x^4 & -y - x^3 & ix^2 & x & -i & 0 \\ -x^5 - 5x^2y & 2xy + x^4 & -iy - ix^3 & -x^2 & ix & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{iL}_{2,1}(\mathbf{x},\mathbf{y}) & -iF_{2,1}(x,y) & 0 & 0 & 0 & 0 \\ -\mathbf{L}_{2,2}(\mathbf{x},\mathbf{y}) & F_{2,2}(x,y) & -iF_{2,1}(x,y) & 0 & 0 & 0 \\ -\mathbf{iL}_{2,3}(\mathbf{x},\mathbf{y}) & iF_{2,3}(x,y) & F_{2,2}(x,y) & -iF_{2,1}(x,y) & 0 & 0 \\ \mathbf{L}_{2,4}(\mathbf{x},\mathbf{y}) & -F_{2,4}(x,y) & iF_{2,3}(x,y) & F_{2,2}(x,y) & -iF_{2,1}(x,y) & 0 \\ -\mathbf{L}_{2,5}(\mathbf{x},\mathbf{y}) & \mathbf{F}_{2,5}(\mathbf{x},\mathbf{y}) & -\mathbf{iF}_{2,4}(\mathbf{x},\mathbf{y}) & -\mathbf{F}_{2,3}(\mathbf{x},\mathbf{y}) & \mathbf{iF}_{2,2}(\mathbf{x},\mathbf{y}) & \mathbf{F}_{2,1}(\mathbf{x},\mathbf{y}) \end{bmatrix}.$$

Theorem 2.6. *Let $p \geq 1$ be an integer, $M_{p,n}$ be an $n \times n$ Hessenberg matrix in* (1.13) *and*

$$\widetilde{M}_{p,n} = \begin{bmatrix} 1 & 0 & 0 \\ & & \\ & -M_{p,n} & 0 \\ & & 1 \end{bmatrix}.$$

Then, $(\widetilde{M}_{p,n})^{-1} =$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ L_{p,1}(x,y) & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ L_{p,n-3}(x,y) & F_{p,n-3}(x,y) & \ddots & \ddots & 0 & 0 \\ L_{p,n-2}(x,y) & F_{p,n-2}(x,y) & F_{p,n-3}(x,y) & \ddots & 0 & 0 \\ L_{p,n-1}(x,y) & F_{p,n-1}(x,y) & F_{p,n-2}(x,y) & \ddots & 1 & 0 \\ L_{p,n}(x,y) & F_{p,n}(x,y) & F_{p,n-1}(x,y) & \cdots & F_{p,2}(x,y) & 1 \end{bmatrix}.$$

Proof. Proof is similar to the proof of Theorem 2.1, by using (1.14). □

Example 2.7. *We obtain some terms of generalized bivariate Fibonacci-*3 *and Lucas-*3 *numbers, by using Theorem 2.6.*

$$\widetilde{M}_{3,8} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 & 0 & 0 & 0 \\ -4y & 0 & 0 & -x & 1 & 0 & 0 \\ 0 & -y & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & -y & 0 & 0 & -x & 1 \end{bmatrix}$$

and inverse of $\widetilde{M}_{3,8}$ is

$$(\widetilde{M}_{3,8})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 & 0 & 0 \\ x^2 & x & 1 & 0 & 0 & 0 & 0 \\ x^3 & x^2 & x & 1 & 0 & 0 & 0 \\ 4y + x^4 & x^3 & x^2 & x & 1 & 0 & 0 \\ 5xy + x^5 & y + x^4 & x^3 & x^2 & x & 1 & 0 \\ x^6 + 6x^2y & 2xy + x^5 & y + x^4 & x^3 & x^2 & x & 1 \end{bmatrix}$$

$$
= \begin{bmatrix}
\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{L}_{3,1}(\mathbf{x},\mathbf{y}) & 1 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{L}_{3,2}(\mathbf{x},\mathbf{y}) & F_{3,2}(x,y) & 1 & 0 & 0 & \vdots & \vdots \\
\mathbf{L}_{3,3}(\mathbf{x},\mathbf{y}) & F_{3,3}(x,y) & F_{3,2}(x,y) & 1 & 0 & 0 & 0 \\
\mathbf{L}_{3,4}(\mathbf{x},\mathbf{y}) & F_{3,4}(x,y) & F_{3,3}(x,y) & F_{3,2}(x,y) & 1 & 0 & 0 \\
\mathbf{L}_{3,5}(\mathbf{x},\mathbf{y}) & F_{3,5}(x,y) & F_{3,4}(x,y) & F_{3,3}(x,y) & F_{3,2}(x,y) & 1 & 0 \\
\mathbf{L}_{3,6}(\mathbf{x},\mathbf{y}) & \mathbf{F}_{3,6}(\mathbf{x},\mathbf{y}) & \mathbf{F}_{3,5}(\mathbf{x},\mathbf{y}) & \mathbf{F}_{3,4}(\mathbf{x},\mathbf{y}) & \mathbf{F}_{3,3}(\mathbf{x},\mathbf{y}) & \mathbf{F}_{3,2}(\mathbf{x},\mathbf{y}) & \mathbf{1}
\end{bmatrix}.
$$

The preceding results allows to obtain terms of more than thirty sequences and polynomials. The following lemma shows this fact for some of these sequences and polynomials.

Lemma 2.8. *[18] $F_{p,n}(x,y)$ and $L_{p,n}(x,y)$ are general form of many sequences and polynomials as,*

x	y	p	$F_{p,n}(x,y)$	$L_{p,n}(x,y)$
x	y	1	*Bivariate Fibonacci poly.* $F_n(x,y)$	*Bivariate Lucas poly.* $L_n(x,y)$
x	1	p	*Fibonacci p-polynomials* $F_{p,n}(x)$	*Lucas $p-$polynomials* $L_{p,n}(x)$
x	1	1	*Fibonacci polynomials* $f_n(x)$	*Lucas polynomials* $l_n(x)$
1	1	p	*Fibonacci p-numbers* $F_p(n)$	*Lucas p-numbers* $L_p(n)$
1	1	1	*Fibonacci numbers* F_n	*Lucas numbers* L_n
$2x$	y	p	*Bivariate Pell p-poly.* $F_{p,n}(2x,y)$	*Bivariate Pell–Lucas p-polynomials*
$2x$	y	1	*Bivariate Pell polynomials* $F_n(2x,y)$	*Bivariate Pell–Lucas poly.* $L_n(2x,y)$
$2x$	1	p	*Pell p-polynomials* $P_{p,n}(x)$	*Pell–Lucas p-polynomials* $Q_{p,n}(x)$
$2x$	1	1	*Pell polynomials* $P_n(x)$	*Pell–Lucas polynomials* $Q_n(x)$
2	1	1	*Pell numbers* P_n	*Pell–Lucas numbers* Q_n
$2x$	-1	1	*Second kind Chebyshev poly.* $U_{n-1}(x)$	*Chebyshev poly. of the first kind*
x	$2y$	p	*Bivariate Jacobsthal p-poly.*	*Bivariate Jacobsthal–Lucas p-poly.*
x	$2y$	1	*Bivariate Jacobsthal Poly.* $F_n(x,2y)$	*Bivariate Jacobsthal–Lucas poly.*
1	$2y$	1	*Jacobsthal Polynomials* $J_n(y)$	*Jacobsthal–Lucas poly.* $j_n(y)$
1	2	1	*Jacobsthal Numbers* J_n	*Jacobsthal–Lucas numbers* j_n

3 Conclusion

In this paper, we obtained terms of generalized bivariate Fibonacci and Lucas p-polynomials. Our results are more general and more comprehensive than the studies on many sequences and polynomials referred in Lemma 2.8 and include them. There are a lot of methods to calculate terms of a linear recursion sequences and polynomials. Some of these methods give only one term, such as Binet formulas, and some of them give successive n terms, as the authors previous studies, whereas this study gives opportunity to calculate many terms of two different linear recursion sequences at a time of calculation. For example successive n terms of Chebyshev polynomials of the first kind and second kind can be calculated, simultaneously, using these results.

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