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On an analogue of Buchstab's identity

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Abstract: In this paper, let p denote a prime. We shall consider sums of the type $\Phi(x,y;f) = \sum_{n\le x, p|n \Rightarrow p>y} f(n)$ and $\psi(x,y;f) = \sum_{n\le x, p|n\Rightarrow p<y} f(n)$ for certain kinds of arithmetical functions f and prove some identities for Φ and ψ which are analogous to the 'so-called' Buchstab identity. As an application, we will prove some formulas for square-free integers.
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1 Introduction

In this article, we will study the analogue of the following identity

$$\Psi(x,y) = \Psi(x,z) - \sum_{y<p\le z} \Psi\left(\frac{x}{p}, p\right), \tag{1}$$

where p denotes any prime, x, y, z are positive real numbers such that $x \ge z \ge y \ge 1$ and $\Psi(x,y)$ is the number of integers up to x whose prime factors are all $\le y$:

$$\Psi(x,y) = \sum_{\substack{n\le x \\ p|n\Rightarrow p\le y}} 1.$$

The above identity (1) is called Buchstab's identity [3]. Several researchers investigated the function $\Psi(x,y)$, including Dickman [6], de Bruijn [4, 5], Hilderbrand [7] and Hilderbrand and Tenenbaum [8]. By using the identity (1), Chebycheff's estimate

$$\pi(x) = \sum_{p \le x} 1 = O\left(\frac{x}{\log x}\right), \tag{2}$$

and Mertens' formula

$$\sum_{p \le x} \frac{1}{p} = \log\log x + A + E_1(x), \quad E_1(x) = O\left(\frac{1}{\log x}\right), \tag{3}$$

we obtain the following formula. For any $\varepsilon > 0$ and $x^\varepsilon < y \le x$,

$$\Psi(x, y) = x\rho(u) + O\left(\frac{x}{\log y}\right),$$

where $u = \log x / \log y$ and the function $\rho(u)$ is defined by

$$\rho(u) = \begin{cases} 1 & (0 \le u \le 1), \\ 1 - \int_1^u \frac{\rho(v-1)}{v} dv & (u \ge 1). \end{cases} \tag{4}$$

This function $\rho(u)$ is called Dickman's function [6]. Substantial progress on the problem of estimating $\Psi(x, y)$ was made by de Bruijn [4]. Similarly, we can consider the following analogue of Buchstab's identity by defining $\Phi(x, y)$ to be the number of integers $n \le x$ all of whose prime factors are greater than y:

$$\Phi(x, y) = \sum_{\substack{n \le x \\ p \mid n \Rightarrow p > y}} 1.$$

For $x \ge z \ge y \ge 1$, we have

$$\Phi(x, y) = \Phi(x, z) + \sum_{y < p \le x} \Phi\left(\frac{x}{p}, p\right) + O\left(\frac{x}{y}\right).$$

This identity helps one to derive an asymptotic formula for $\Phi(x, y)$. For any $\varepsilon > 0$ and $x^\varepsilon < y \le x$, using the prime number theorem of the form

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right) \tag{5}$$

one can get

$$\Phi(x, y) = \frac{x\omega(u) - y}{\log y} + O\left(\frac{x}{\log^2 y}\right). \tag{6}$$

Here $\omega(u)$ as the function $\rho(u)$ before is defined recursively:

$$\omega(u) = \begin{cases} \frac{1}{u} & (1 \le u \le 2), \\ \frac{1}{u} + \frac{1}{u}\int_1^{u-1} \omega(v) dv & (u \ge 2). \end{cases} \tag{7}$$

Some analogues of $\Psi(x, y)$ and $\Phi(x, y)$ are considered by Alladi [1, 2] and Ivić [9]. Being motivated by these studies we shall consider analogues of Buchstab's identity and deduce some results concerned with square-free integers.

Now we shall define three summatory functions concerned with f as follows:

Definition 1.1. *Let $x \geq y \geq 1$ and for an arithmetical function f,we define*

$$M(x;f) = \sum_{n \leq x} f(n),$$
$$\psi(x,y;f) = \sum_{\substack{n \leq x \\ p|n \Rightarrow p<y}} f(n),$$
$$\Phi(x,y;f) = \sum_{\substack{n \leq x \\ p|n \Rightarrow p>y}} f(n).$$

Remark 1. *If $y \geq x$, then clearly,*

$$\psi(x,y;f) = M(x;f) + O(|f(x)|) \quad \text{and} \quad \Phi(x,y;f) = 1.$$

Now we add two restrictions on f:

$$\begin{cases} (A) \quad f \text{ is multiplicative,} \\ (B) \quad f(p^m) = 0 \text{ for any prime and positive integer } m \geq 2. \end{cases} \tag{8}$$

Under these assumptions, we obtain analogues of Buchstab's identity (see, e.g., Tenenbaum [10, p. 365, p. 398]).

Theorem 1.2. *Keeping the notations as above and for $x \geq z \geq y \geq 1$, we have*

$$\psi(x;y;f) = 1 + \sum_{p<y} f(p)\psi\left(\frac{x}{p}, p; f\right), \tag{9}$$

$$\psi(x,y;f) = \psi(x,z;f) - \sum_{y \leq p < z} f(p)\psi\left(\frac{x}{p}, p; f\right), \tag{10}$$

$$\Phi(x,y;f) = 1 + \sum_{y<p\leq x} f(p)\Phi\left(\frac{x}{p}, p; f\right), \tag{11}$$

$$\Phi(x,y;f) = \Phi(x,z;f) + \sum_{y<p\leq z} f(p)\Phi\left(\frac{x}{p}, p; f\right). \tag{12}$$

We shall apply the above formulas (11) and (12) to the arithmetical functions μ, μ^2 and μ/N, where μ is the Möbius function and $N(n) = n$. These three functions satisfy the required conditions (8).

For example we have

Theorem 1.3. *For $x^{\varepsilon} < y \leq x$, then*

$$\Phi\left(x, y; \frac{\mu}{N}\right) = \rho(u) + O\left(\frac{1}{\log y}\right), \tag{13}$$

where $u = \log x / \log y$ and $\rho(u)$ is the Dickman function.

Corollary 1.4. *For any* $\alpha > 1$

$$\lim_{x\to\infty} \Phi\left(x, x^{1/\alpha}; \frac{\mu}{N}\right) = \rho(\alpha).$$

The left hand side of the above is the sum $\sum_{n=1}^{\infty} \mu(n)/n$ with the condition $p|n \Rightarrow p > y$.

Remark 2. *The prime number theorem* $\pi(x) \sim x/\log x$ *is equivalent to* $\sum_{n=1}^{\infty} \mu(n)/n = 0$. *Addition of the condition prime factors greater than* y*, makes the Dickman function to appear in the formula .*

As another application of Theorem 1.2, we shall define

$$\mathcal{Q}(x,y) = \Phi(x,y;\mu^2) = \sum_{\substack{n\le x \\ p|n\Rightarrow p>y}} \mu^2(n), \tag{14}$$

$$\mathcal{R}(x,y) = \Phi(x,y;\mu) = \sum_{\substack{n\le x \\ p|n\Rightarrow p>y}} \mu(n). \tag{15}$$

By formulas (11) and (12) we have

Theorem 1.5. *For* $x^{\varepsilon} < y \le x$*, by the prime number theorem of the form (5), we have*

$$\mathcal{Q}(x,y) = \frac{x\omega(u) - y}{\log y} + O\left(\frac{x}{\log^2 y}\right), \tag{16}$$

$$\mathcal{R}(x,y) = \frac{x\rho'(u) + y}{\log y} + O\left(\frac{x}{\log^2 y}\right), \tag{17}$$

where $u = \log x/\log y$*,* $\omega(u)$ *is the Buchstab function (see (7)) and* $\rho'(u)$ *is the derivative of* $\rho(u)$*.*

Trivially, when $y \ge x \ge 1$ *we see* $\mathcal{Q}(x,y) = \mathcal{R}(x,y) = 1$.

Remark 3. *In [1, p. 87, Theorem 1], by (5) Alladi studied the asymptotic formula for* $\mathcal{R}(x,y)$*. His result shows the error term of (17) is* $O(x \cdot u^2/\log^2 y)$ *uniformly for* $x \ge y \ge 2$*. In the final section of this paper, by using the prime number theorem of the form*

$$\pi(x) = li(x) + O\left(x \exp\left(-c\sqrt{\log x}\right)\right), \tag{18}$$

(where $li(x) = \int_2^x \frac{dt}{\log t}$*,* $c > 0$ *is a constant), we shall consider the above theorem. See Theorem 4.1 below.*

2 Proof of Theorem 1.2 and an application

First of all we shall prove Theorem 1.2. Let f be an arithmetical function satisfying (8). By Definition 1.1 we have the assertion (9) as follows

$$\psi(x,y;f) = 1 + \sum_{p<y} \sum_{\substack{pm\le x,\, p\nmid m \\ q|m\Rightarrow q<p}} f(pm) = 1 + \sum_{p<y} f(p) \sum_{\substack{pm\le x \\ q|m\Rightarrow q<p}} f(m).$$

The second assertion (10) comes from (9) at once.

By the argument similar to the above, we have the formula (11)

$$\Phi(x,y;f)=1+\sum_{y<p\le x}\sum_{\substack{pm\le x,\, p\nmid m\\ q|m\Rightarrow q>p}} f(pm)=1+\sum_{y<p\le x} f(p)\sum_{\substack{m\le x/p\\ q|m\Rightarrow q>p}} f(m).$$

Form (11), we can obtain the identity (12) easily.

Let $u=\log x/\log y$. As an application of Theorem 1.2, we shall prove Theorem 1.3.

Proof. Let us assume $u\in(1,2]$, then by Eratosthenes' sieve and (3) we observe that

$$\Phi\left(x,y;\frac{\mu}{N}\right)=1-\sum_{y<p\le x}\frac{1}{p}=1-\log u+O\left(\frac{1}{\log y}\right).$$

Now we assume that the formula (13) is true for $u\in(1,2],(2,3],\dots,(K-1,K]$. In the case of $u\in(K,K+1]$, we put in (12), $y=x^{1/u}$ and $z=x^{1/K}$ then

$$\Phi\left(x,x^{1/u};\frac{\mu}{N}\right)=\rho(K)+O\left(\frac{1}{\log y}\right)-\sum_{x^{1/u}<p\le x^{1/K}}\frac{1}{p}\Phi\left(\frac{x}{p},p;\frac{\mu}{N}\right).$$

In the above sum, since $\frac{\log\frac{x}{p}}{\log p}\le K$ we shall apply our assumption to get

$$\begin{aligned}&\sum_{x^{1/u}<p\le x^{1/K}}\left\{\frac{1}{p}\rho\left(\frac{\log x}{\log p}-1\right)+O\left(\frac{1}{p\log p}\right)\right\}\\&=\sum_{x^{1/u}<p\le x^{1/K}}\frac{1}{p}\rho\left(\frac{\log x}{\log p}-1\right)+O\left(\frac{1}{\log y}\right)\\&=\int_{x^{1/u}}^{x^{1/K}}\rho\left(\frac{\log x}{\log w}-1\right)d\log\log w+\int_{x^{1/u}}^{x^{1/K}}\rho\left(\frac{\log x}{\log w}-1\right)dE_1(w)\\&\qquad+O\left(\frac{1}{\log y}\right)\\&:=A+B+O\left(\frac{1}{\log y}\right)\quad\text{(say)},\end{aligned}$$

where $E_1(\cdot)$ is the same error term in (3).

Putting $v=\log x/\log w$ we have $A=\int_K^u\rho(v-1)v^{-1}dv$.

Moreover since ρ, ρ' are bounded ([10, p. 366]) we see $B=O(1/\log y)$.

Therefore, for $u\in(K,K+1]$ we obtain

$$\begin{aligned}\Phi\left(x,x^{1/u};\frac{\mu}{N}\right)&=\rho(K)-\int_K^u\frac{\rho(v-1)}{v}dv+O\left(\frac{1}{\log y}\right)\\&=\rho(u)+O\left(\frac{1}{\log y}\right).\end{aligned}$$

From this, we observe that the assertion (13) is valid for $x^{\varepsilon}<y\le x$. □

3 On square-free integers

In this section, we shall consider an application of (11) and (12) on square-free numbers. So we shall prove Theorem 1.5.

Proof. (Proof of Theorem 1.5.) We will prove the latter formula (17) only. In fact, we can prove the previous formula (16) by a similar method.

First we shall notice that

$$\rho'(u) = \begin{cases} -\frac{1}{u} & (1 \leq u \leq 2), \\ -\frac{1}{u} - \frac{1}{u}\int_2^u \rho'(v-1)dv & (u \geq 2). \end{cases} \tag{19}$$

By (11), Eratosthenes' sieve, the prime number theorem (5) and (19) we have

$$\begin{aligned} \mathcal{R}(x,y) &= 1 + \sum_{y<p\leq x} \mu(p) = 1 - \pi(x) + \pi(y) \\ &= \frac{x\rho'(u)+y}{\log y} + O\left(\frac{x}{\log^2 y}\right) \quad \textit{for } u \in (1,2]\ (\textit{or } \sqrt{x} \leq y < x). \end{aligned} \tag{20}$$

For $u \in (2,3]$, by (12) with $f = \mu$, $y = x^{1/u}$, and $z = x^{1/2}$ we have

$$\mathcal{R}(x,y) = \mathcal{R}(x,x^{1/2}) - \sum_{x^{1/3}<p\leq x^{1/2}} \mathcal{R}\left(\frac{x}{p},p\right) + O\left(\frac{x}{\log^2 y}\right).$$

Since $(\log x/p)/\log p = \log x/\log p - 1 \leq 2$, using (20) we can show that (17) is valid for $u \in (2,3]$ (the method is similar to the generalized argument just below).

Here we assume the formula (17) is true for $u \in (3,4]$, $(4,5], \dots, (N-1,N]$ $(N \geq 3$). We shall consider it for $u \in (N, N+1]$ and take $f = \mu$, $y = x^{1/u}$ and $z = x^{1/N}$ in (12), then we have

$$\begin{aligned} \mathcal{R}(x,y) = &\frac{x\rho'(N)+x^{1/N}}{\log x^{1/N}} + \frac{y}{\log y} - \frac{y}{\log y} \\ &- \sum_{x^{1/u}<p\leq x^{1/N}} \mathcal{R}\left(\frac{x}{p},p\right) + O\left(\frac{x}{\log^2 y}\right). \end{aligned}$$

Since $\frac{\log\frac{x}{p}}{\log p} = \frac{\log x}{\log p} - 1 \leq N$ we can get

$$\begin{aligned} \sum_{x^{1/u}<p\leq x^{1/N}} \mathcal{R}\left(\frac{x}{p},p\right) &= x\sum_{x^{1/u}<p\leq x^{1/N}} \frac{\rho'\left(\frac{\log x}{\log p}-1\right)}{p\log p} + \sum_{x^{1/u}<p\leq x^{1/N}} \frac{p}{\log p} \\ &\quad + O\left(x\sum_{x^{1/u}<p\leq x^{1/N}} \frac{1}{p\log^2 p}\right) \\ &=: xA + B + C \quad \text{(say)}. \end{aligned}$$

Using (5) and (3) we have $B, C \ll x/\log^2 y$ respectively. Also by (3) we see

$$A = \int_{x^{1/u}}^{x^{1/N}} \frac{\rho'\left(\frac{\log x}{\log w}-1\right)}{\log w} d\log\log w + \int_{x^{1/u}}^{x^{1/N}} \frac{\rho'\left(\frac{\log x}{\log w}-1\right)}{\log w} dE_1(w).$$

By putting $v = \log x/\log w$, the former integral is

$$-\frac{1}{\log x}\int_u^N \rho'(v-1)dv,$$

and the latter integral is

$$\left[\frac{\rho'\left(\frac{\log x}{\log w}-1\right)}{\log w}E_1(w)\right]_{x^{1/u}}^{x^{1/N}} \tag{21}$$

$$+\log x\int_{x^{1/u}}^{x^{1/N}}\frac{\rho''\left(\frac{\log x}{\log w}-1\right)}{w\log^3 w}E_1(w)dw+\int_{x^{1/u}}^{x^{1/N}}\frac{\rho'\left(\frac{\log x}{\log w}-1\right)}{w\log^2 w}E_1(w)dw.$$

Since ρ' is bounded and $E_1(w) = O(1/\log w)$ the first part of (21) is estimated as $O(1/\log^2 y)$. Moreover, since $1/\log y = O(N/\log x)$ and $\log((N+1)/N) = O(1/N)$ we can estimate the middle and last parts of (21) as $O(1/\log^2 y)$ respectively. Hence for $u \in (N, N+1]$ we obtain

$$\begin{aligned}
\mathcal{R}(x,y) &= \frac{x\rho'(N)}{\log x^{1/N}}+\frac{x}{\log x}\int_u^N \rho'(v-1)dv+\frac{y}{\log y}+O\left(\frac{x}{\log^2 y}\right)\\
&= \frac{x}{\log y}\left(\frac{(\log y)\rho'(N)}{\log x^{1/N}}+\frac{\log y}{\log x}\int_u^N \rho'(v-1)dv\right)+\frac{y}{\log y}+O\left(\frac{x}{\log^2 y}\right)\\
&= \frac{x}{\log y}\left(\frac{N}{u}\left(-\frac{1}{N}-\frac{1}{N}\int_2^N \rho'(v-1)\right)-\frac{1}{u}\int_N^u \rho'(v-1)dv\right)\\
&\quad +\frac{y}{\log y}+O\left(\frac{x}{\log^2 y}\right)\\
&= \frac{x\rho'(u)+y}{\log y}+O\left(\frac{x}{\log^2 y}\right).
\end{aligned}$$

This shows that the formula (17) is valid for $x^{\varepsilon} < y \le x$. □

We shall observe the numbers of two kinds of restricted square-free integers, based on Theorem 1.5.

Definition 3.1. *Let m be a positive square-free integer and $\nu(m)$ the number of distinct prime factors of m. For $x \ge y \ge 1$ we define the following counting functions:*

$$\mathcal{Q}_{\text{even}}(x,y) := \sum_{\substack{m\le x,\ \nu(m):\ \text{even}\\ p|m\Rightarrow p>y}} 1 = \sum_{\substack{n\le x\\ p|n\Rightarrow p>y}} \frac{\mu^2(n)+\mu(n)}{2},$$

$$\mathcal{Q}_{\text{odd}}(x,y) := \sum_{\substack{m\le x,\ \nu(m):\ \text{odd}\\ p|m\Rightarrow p>y}} 1 = \sum_{\substack{n\le x\\ p|n\Rightarrow p>y}} \frac{\mu^2(n)-\mu(n)}{2},$$

where we regard 1 as $\nu(1)$ is even.

If we use $M(x;\mu) = o(x)$ (which is equivalent to the prime number theorem in the form $\pi(x) \sim x/\log x$) and $M(x;\mu^2) = \frac{6}{\pi^2}x + O(\sqrt{x})$, then we have easily

$$\mathcal{Q}_{\text{even}}(x,1) = \frac{3}{\pi^2}x + o(x) \text{ and } \mathcal{Q}_{\text{odd}}(x,1) = \frac{3}{\pi^2}x + o(x).$$

However, if y is large by Theorem 1.5 we get the following corollary.

Corollary 3.2. *For $x^\varepsilon < y \le x$ and $u = \frac{\log x}{\log y}$,*

$$\mathcal{Q}_{\text{even}}(x,y) = \frac{x}{\log y}\left(\frac{\omega(u)+\rho'(u)}{2}\right) + O\left(\frac{x}{\log^2 y}\right),$$
$$\mathcal{Q}_{\text{odd}}(x,y) = \frac{x}{\log y}\left(\frac{\omega(u)-\rho'(u)}{2}\right) - \frac{y}{\log y} + O\left(\frac{x}{\log^2 y}\right).$$

4 Remarks

In this final section, following [10, p. 400, Theorem 3] we shall attempt to extend the range $x^\varepsilon < y \le x$ in Theorem 1.5. By the prime number theorem of the form (18) we have

$$\sum_{p\le x}\frac{1}{p} = \log\log x + A + O\left(\exp\left(-B\sqrt{\log x}\right)\right), \tag{22}$$

where A is a constant and B is a positive constant. With the help of (18) and (22) we obtain the following.

Theorem 4.1. *Uniformly for $x \ge y \ge 2$, we have*

$$\mathcal{Q}(x,y) = \frac{x\omega(u) - y}{\log y} + O\left(\frac{x}{\log^2 y}\right), \tag{23}$$
$$\mathcal{R}(x,y) = \frac{x\rho'(u) + y}{\log y} + O\left(\frac{x}{\log^2 y}\right), \tag{24}$$

where the notation is same as the above.

Proof. Since trivially $\mathcal{Q}(x,y)$ and $\mathcal{R}(x,y) = O(x)$, so if y is bounded then (23) and (24) are obviously true. So we assume that $y \ge y_0$, where y_0 is a sufficiently large constant. In addition, we may also assume that $u > 3$ in fact we have already proved Theorem 1.5. Let $\Delta(x,y)$ be the function implicitly defined by the formula

$$\mathcal{R}(x,y) = \frac{x}{\log y}\left(\rho'(u) + \frac{\Delta(x,y)}{\log y}\right). \tag{25}$$

We shall establish by induction on integers $k \ge 3$, that the quantity

$$\Delta_k := \sup\left\{|\Delta(x,y)| \,|\, y \ge y_0,\ 2 < u \le k\right\}.$$

is finite and bounded independently of k. By Theorem 1.5 we see that $\Delta_3 < +\infty$. Let $k \ge 3$ be such that $\Delta_k < +\infty$. We shall consider the case $y \ge y_0$ and $2 < u \le k+1$. By the identity (12) with $f = \mu$ and $z = \sqrt{x}$ and (25) we observe that

$$\mathcal{R}(x,y) = \mathcal{R}\left(x,\sqrt{x}\right) - \sum_{y<p\le\sqrt{x}}\frac{x}{p\log p}\left\{\rho'\left(\frac{\log x}{\log p} - 1\right) + \frac{\theta_p\Delta_k}{\log p}\right\}$$

with $\theta_p = \theta_p(x) \in [-1,1]$. By (5) we have

$$\mathcal{R}\left(x,\sqrt{x}\right) = -\frac{x}{\log x} + O\left(\frac{x}{\log^2 y}\right),$$

By (22) for any sufficiently large $y \geq y_0$ we have

$$\sum_{p>y} \frac{1}{p\log^2 p} = \frac{\frac{1}{2} + O\left(\exp\left(-B\sqrt{\log x}\right)\right)}{\log^2 y} \leq \frac{3}{4\log^2 y},$$
$$H(v) = \sum_{x^{1/v}<p\leq\sqrt{x}} \frac{1}{p} = \log\frac{v}{2} + O\left(\exp\left(-B\sqrt{\log x^{1/v}}\right)\right). \tag{26}$$

By the Stieltjes integral with (26) we see that

$$\sum_{y<p\leq\sqrt{x}} \frac{\rho'\left(\frac{\log x}{\log p} - 1\right)}{p\log p} = \frac{1}{\log x}\int_2^u \rho'(v-1)dv + O\left(\frac{u\exp\left(-B\sqrt{\log y}\right)}{\log x}\right)$$
$$= \frac{-u\rho'(u)+1}{\log x} + O\left(\frac{u\exp\left(-B\sqrt{\log y}\right)}{\log x}\right).$$

Collecting the above calculations we have

$$\mathcal{R}(x,y) = \frac{x}{\log y}\left(\rho'(u) + O\left(\exp\left(-B\sqrt{\log y}\right)\right)\right) + \frac{x\left(\theta\Delta_k + O(1)\right)}{\log^2 y}. \tag{27}$$

By (27) we see that $\Delta_{k+1} \leq 4C$ with a constant $C > 0$. It completes the proof of (24). By a similar argument we may prove (23). □

We have also

Corollary 4.2. *Uniformly for $x \geq y \geq 2$ we have*

$$\mathcal{Q}_{\text{even}}(x,y) = \frac{x}{\log y}\left(\frac{\omega(u)+\rho'(u)}{2}\right) + O\left(\frac{x}{\log^2 y}\right),$$
$$\mathcal{Q}_{\text{odd}}(x,y) = \frac{x}{\log y}\left(\frac{\omega(u)-\rho'(u)}{2}\right) - \frac{y}{\log y} + O\left(\frac{x}{\log^2 y}\right).$$

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