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A note on Dedekind's arithmetical function

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Abstract: We point out that an inequality published recently in [1] is a particular case of a general result from [4]. By another method, a refinement is offered, too. Related inequalities are also proved.
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1 Introduction

Let $\psi(n)$ be the Dedekind arithmetic function, defined by $\psi(1) = 1$, and $\psi(n) = n \prod_{i=1}^{r} \left(1 + \frac{1}{p_i}\right)$ for $n > 1$. Here p_i denote the distinct prime divisors of n. In the recent paper [1], the following inequality is stated:

$$(\psi(ab))^k \geq \psi(a^k)\psi(b^k), \text{ for any } a, b \geq 1 \text{ and } k \geq 2 \tag{1}$$

We note that, (1) is a particular case of a result from our published paper (and arXiv Preprint) [4] (see Theorem 7 of [4]). Namely, for any arithmetical function f satisfying

$$(f(ab))^k \geq f(a^k)f(b^k); a, b \geq 1, k \geq 2 \tag{2}$$

are called k-*super multiplicative*, and when (2) holds with reversed inequality (i.e. "$\leq$"), the k-*submultiplicative* functions.

In [4] it is proved that, if f is a multiplicative function, with $f(1) = 1$, and for any integers $x, y \geq 0$, and $k \geq 1$, integer, p prime, one has

$$(f(p^{x+y}))^k \geq (f(p^{kx})f(p^{ky}), \tag{3}$$

then (2) holds true, i.e., the function f is k-super multiplicative.

It is easy to see that, when $f(n) = \psi(n)$, inequality (3) becomes

$$\left(1 + \frac{1}{p}\right)^k \geq \left(1 + \frac{1}{p}\right)^2$$

so clearly, (3) is true. We note that, for $f(n) = \varphi(n)$ (i.e. Euler's totient), in [4] (and arXiv Preprint) is proved the reverse inequality, but there are considered also many other particular cases.

2 Main results

The following refinement of (1) holds true:

Theorem 1. *For any integers $a, b \geq 1$ and $k \geq 2$ one has*

$$(\psi(ab))^k \geq ((ab)^{k-2}) \cdot (\psi(ab))^2 \geq \psi(a^k)\psi(b^k) \tag{4}$$

Proof. The following known properties of the function $\psi(n)$ will be applied:

$$\psi(n^k) = n^{k-1}\psi(n); \tag{5}$$

$$\psi(nm) \geq n\psi(m) \tag{6}$$

for any integrs $n, k, m \geq 1$. For proofs, see e.g. [2], [3].

Now, the first relation of (4) may be rewritten as

$$\left(\frac{\psi(ab)}{ab}\right)^k \geq \left(\frac{\psi(ab)}{ab}\right)^2 \tag{7}$$

which is true, as $k \geq 2$ and by $\psi(n) \geq n$ one has $\frac{\psi(ab)}{ab} \geq 1$. The second inequality of (4) may be rewritten as

$$(\psi(ab))^2 \geq ab\psi(a)\psi(b) \tag{8}$$

This is a consequence of relation (6) applied twice: $\psi(ab) \geq a\psi(b)$ and $\psi(ab) \geq b\psi(a)$. The proof of Theorem 1 is finished. □

Theorem 2. *For any integers $n \geq k \geq 2$ and $a, b \geq 1$ one has*

$$(\psi(ab))^k \geq (ab)^{n-k}\psi(a^n)\psi(b^n) \tag{9}$$

Proof. Applying inequality (2) for $a = x^n$ and $b = y^n$ ($x, y \geq 1$ integers), and by taking into account of (5), after easy computations we get

$$(\psi(xy))^k \geq (xy)^{n-k}\psi(x^n)\psi(y^n),$$

which is in fact relation (9) with x and y in place of a and b, respectively. □

Remark. For $n = k$, relation (9) implies inequality (1).

Theorem 3. *For any integers $a, b \geq 1$ and $k \geq 2$ one has*

$$(ab)^{k-1}\frac{\psi(ab)}{\psi(a)\psi(b)} \leq (ab)^{k-1} \leq \frac{(\psi(ab))^k}{\psi(a)\psi(b)} \leq (\psi(a)\psi(b))^{k-1} \tag{10}$$

Proof. Applying the property $\psi(ab) \leq \psi(a)\psi(b)$ (see [2, 3]), and relations (2) and (5), we can write $(ab)^{k-1}\psi(a)\psi(b) \leq (\psi(ab))^k \leq (\psi(a))^k(\psi(b))^k$, which immediately give the last two inequalities of (10). The first relation of (10) is in fact the above stated property (by reducing with $(ab)^{k-1}$). $\square$

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