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A generalization of Ivan Prodanov's inequality

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In memory of my teacher
Prof. Ivan Prodanov (1935–1985)

Abstract: A generalization of Prof. Ivan Prodanov's inequality is formulated and proved. As partial case, an inequality discussed in D. Mitrinović and M. Popadić's book [2] is obtained.
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1 Introduction

The paper was written in 1995 when in Faculty of Mathematics at Sofia University a Special Session devoted to the memory of Prof. Ivan Prodanov (1935–1985) was organized ten years after his death. It should have had the following bibliographic data:

> Atanassov, K. (1994) A generalization of Ivan Prodanov's inequality. *Annuaire de l'Universite de Sofia "St. Kliment Ohridski", Faculte de Mathematiques et Informatique, Livre 1 – Mathematiques,* Tome 88, 117–120.

but for already 20 years, this issue of the "Annuaire" had not been printed. So, the author decided to publish the text in its original form in memory of Prof. Prodanov, on the 30-th year after his death.

2 Main results

Let $\{a_i\}_{i=1}^{n}$ be a sequence of real numbers, where n is a fixed natural number, and let p and q be fixed integers, for which $p \geq n-1$ and $1-p \leq q \leq \frac{p+1}{n}$. Let $[x]$ be the integer part of the real number x.

In the paper it will be proved the following theorem.

Theorem: For every natural number n it holds that

$$p.\sum_{i=1}^{n}[a_i] + q.[\sum_{i=1}^{n} a_i] \leq \sum_{i=1}^{n}[(p+q).a_i]. \tag{1}$$

Proof: From $[a_i] \leq a_i < [a_i]+1$ it follows that for every i $(1 \leq i \leq n)$ there is a natural number s_i such that

$$[a_i] + \frac{s_i}{p+q} \leq a_i < [a_i] + \frac{s_i+1}{p+q}, \tag{2}$$

i.e.,

$$(p+q).[a_i] + s_i \leq (p+q).a_i < (p+q).[a_i] + s_i + 1.$$

Therefore,

$$[(p+q).a_i] = (p+q).[a_i] + s_i$$

and

$$(p+q).\sum_{i=1}^{n}[a_i] + S \leq (p+q).\sum_{i=1}^{n} a_i < (p+q).\sum_{i=1}^{n}[a_i] + S + n,$$

where

$$S = \sum_{i=1}^{n} s_i.$$

Then,

$$\begin{aligned} A &\equiv \sum_{i=1}^{n}[(p+q).a_i] - p.\sum_{i=1}^{n}[a_i] - q.[\sum_{i=1}^{n} a_i] \\ &= \sum_{i=1}^{n}(p+q).[a_i] + S - p.\sum_{i=1}^{n}[a_i] - q.[\sum_{i=1}^{n} a_i] \\ &= q.\sum_{i=1}^{n}[a_i] + S - q.[\sum_{i=1}^{n} a_i]. \end{aligned} \tag{3}$$

Obviously, if $p-1 \leq q \leq 0$, then $A \geq 0$, because

$$[\sum_{i=1}^{n} a_i] \geq \sum_{i=1}^{n}[a_i].$$

Let us assume below that $0 < q \leq \frac{p+1}{n}$. From (2) it follows that

$$\sum_{i=1}^{n}[a_i] + \frac{S}{p+q} \leq \sum_{i=1}^{n} a_i < \sum_{i=1}^{n}[a_i] + \frac{S+n}{p+q}, \tag{4}$$

from where (since function [.] is monotonous) it follows that

$$\sum_{i=1}^{n}[a_i]+[\frac{S}{p+q}]\leq[\sum_{i=1}^{n}a_i]\leq\sum_{i=1}^{n}[a_i]+[\frac{S}{p+q}]+1,$$

because

$$[\frac{S+n}{p+q}]\leq[\frac{S+p+q}{p+q}]=[\frac{S}{p+q}]+1.$$

There are two cases for $[\sum\limits_{i=1}^{n}a_i]$:

(a) $[\sum\limits_{i=1}^{n}a_i]=\sum\limits_{i=1}^{n}[a_i]+[\frac{S}{p+q}]$,

(b) $[\sum\limits_{i=1}^{n}a_i]=\sum\limits_{i=1}^{n}[a_i]+[\frac{S}{p+q}]+1$.

If case (a) is valid, then from (3) it holds that

$$A=S-q.[\frac{S}{p+q}]\geq 0.$$

If case (b) is valid, then from (4) it follows that

$$\sum_{i=1}^{n}[a_i]+[\frac{S}{p+q}]+1=[\sum_{i=1}^{n}a_i]\leq\sum_{i=1}^{n}a_i<\sum_{i=1}^{n}[a_i]+\frac{S+n}{p+q},$$

i.e.

$$(p+q).[\frac{S}{p+q}]+(p+q)<S+n.$$

Therefore,

$$(p+q).[\frac{S}{p+q}]+(p+q)+1\leq S+n,$$

because all members of the inequality are integers. Hence,

$$[\frac{S}{p+q}]\leq\frac{S+n-1}{p+q}-1$$

and from (b) it holds that

$$[\sum_{i=1}^{n}a_i]\leq\sum_{i=1}^{n}[a_i]+\frac{S+n-1}{p+q}.$$

From (3) it follows that

$$A\geq q.\sum_{i=1}^{n}[a_i]+S-q.(\sum_{i=1}^{n}[a_i]+\frac{S+n-1}{p+q})=S-\frac{q.(S+n-1)}{p+q}.$$

But

$$\frac{1}{n}-\frac{q}{p+q}=\frac{1}{n}-\frac{1}{1+\frac{p}{q}}\geq\frac{1}{n}-\frac{p+1}{p+p.n+1}=\frac{p+1-n}{n.(p+p.n+1)}\geq 0$$

and, therefore,

$$A \geq S - \frac{1}{n}.(S + n - 1) = \frac{n-1}{n}.(S - 1).$$

If $S = 0$, then for every i $(1 \leq i \leq n)$ from (2) it holds that

$$a_i < [a_i] + \frac{1}{p+q},$$

and, hence,

$$\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} [a_i] + \frac{n}{p+q} \leq \sum_{i=1}^{n} [a_i] + 1.$$

Therefore,

$$[\sum_{i=1}^{n} a_i] = \sum_{i=1}^{n} [a_i],$$

which coincides with case (a) for $S = 0$. If $S \geq 1$, then, obviously, $A \geq 0$.

This completes the proof of the theorem. □

When $p = n - 1$ and $q = 1$, we obtain the following inequality (see [1]) of Prof. Ivan Prodanov, who taught me mathematical analysis in the Faculty of Mathematics at Sofia University

$$(n-1).\sum_{i=1}^{n} [a_i] + [\sum_{i=1}^{n} a_i] \leq \sum_{i=1}^{n} [n.a_i].$$

On the other hand, the last inequality is extension of inequality 1.48 from [2].

References

[1] Some of Ivan Prodanov's problems (1985) *Matematika*, 26(10), 12–16 (in Bulgarian).

[2] Mitrinović, D. & M. Popadić (1978) *Inequalities in Number Theory.* Niś, Univ. of Niś.