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On some q-Pascal's like triangles

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Abstract: We consider q-analogs of Pascal's like triangles, which were studied by Atanassov in a series of papers.
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Atanassov [2, 3] introduced and studied new types of Pascal's like triangles, which were partly motivated by [4, 6, 7]. In a subsequent series of papers appearing in the same journal, Atanassov derived further properties of these triangles and showed connections to various well-known sequences of numbers. In this paper, we study q-analogs of these new types of Pascal's like triangles as described in [2, 3]. As a paradigmatic example, we construct the following infinite q-triangle

$$\begin{array}{ccccccc} & & & \mathbf{1} & & & \\ & & 1 & \mathbf{q+1} & 1 & & \\ & 1 & q+1 & (\mathbf{q^2+1})(\mathbf{q+1}) & q+1 & 1 & \\ 1 \quad q+1 & (q^2+1)(q+1) & (\mathbf{q^3+1})(\mathbf{q^2+1})(\mathbf{q+1}) & (q^2+1)(q+1) & q+1 \quad 1 \end{array}$$

which is a q-analog of the first triangle given in [2]. Note that the ith element in the middle column is given by $\prod_{k=1}^{i}(q^k+1) = \prod_{k=0}^{i-1}(1+q^{k+1})$, for $i \geq 0$, and reduces to 2^i for $q \to 1$.

In the general case, let the elements of the infinite q-triangle be

$$\begin{array}{ccccccc} & & & a_{00} & & & \\ & & a_{10} & a_{11} & a_{12} & & \\ & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{array}$$

where a_{ij} are arbitrary real (or complex) numbers, and for every $i = 0, 1, 2, \ldots$ and

- for every j such that $1 \leq j \leq i$, $a_{ij} = q^j a_{i(j-1)} + a_{(i-1)(j-1)}$;
- for every j such that $i \leq j \leq 2i-1$, $a_{ij} = q^{2i-j} a_{i(j+1)} + a_{(i-1)(j-1)}$.

Clearly, for $q \to 1$, this recurrence relations reduce to those of the Pascal's like triangles [2, 3]. As in the undeformed case studied by Atanassov, the triangle is symmetric around its middle column, and one may regard the elements a_{i0} ($i \geq 0$) on the left (and right) boundary diagonal as "initial values", determining the entire q-triangle through the defining recurrences.

We now give an expression for an arbitrary element a_{ij} in terms of the elements a_{i0}. It is a q-analog of[1] [2, Lemma 2].

Proposition 1. *Let the infinite sequence $\{c_i\}_{i\geq 0}$ of arbitrary real (or complex) numbers be given, and let, for every $i \geq 0$, $a_{i0} = a_{i(2i)} = c_i$. Then one has, for all $1 \leq j \leq i$, that*

$$a_{ij} = a_{i(2i-j)} = \sum_{k=0}^{j} q^{\binom{k+1}{2}} \binom{j}{k}_q c_{i-j+k}.$$

In particular, the elements of the middle column are given by

$$a_{ii} = \sum_{k=0}^{i} q^{\binom{k+1}{2}} \binom{i}{k}_q c_k.$$

Now, we present two examples for the above proposition. To do that, we use the fact that

$$\prod_{k=0}^{n-1} (1 + q^k t) = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k}_q t^k. \tag{1}$$

Example 2. *Let us choose $c_i = q^{mi}$, that is, $a_{i0} = a_{i(2i)} = q^{mi}$. then the middle column in the q-triangle is given by*

$$a_{ii} = \sum_{k=0}^{i} q^{\binom{k}{2}} \binom{i}{k}_q q^{(m+1)k} = \prod_{k=0}^{i-1} (1 + q^{k+m+1}).$$

In the special case $m = 0$ one has $c_i = 1$, that is, $a_{i0} = a_{i(2i)} = 1$, and one recovers the q-triangle displayed at the beginning.

Example 3. *If we set $a_{i0} = a_{i(2i)} = [i+1]_q = \frac{1-q^{i+1}}{1-q}$, then the middle column in the q-triangle is given by*

$$\begin{aligned} a_{ii} &= \sum_{k=0}^{i} q^{\binom{k}{2}} \binom{i}{k}_q q^k + q^2 [i]_q \sum_{k=0}^{i-1} q^{\binom{k}{2}} \binom{i-1}{k}_q q^{2k} \\ &= \prod_{k=0}^{i-1} (1 + q^{k+1}) + q^2 [i]_q \prod_{k=0}^{i-2} (1 + q^{k+2}). \end{aligned}$$

[1]Note, however, that there is a typo in [2, Lemma 2].

By some straightforward manipulations, one derives from this formula that

$$a_{(i+1)(i+1)} = (1 + q^{i+1})a_{ii} + (1 + q^{i+1})q^{i+2} \prod_{k=0}^{i-2}(1 + q^{k+2}).$$

For $q \to 1$*, one obtains for the undeformed triangle associated to* $a_{i0} = a_{i(2i)} = i + 1$ *that* $a_{(i+1)(i+1)} = 2a_{ii} + 2^i$*, as observed by Atanassov (last triangle in [2], note that the shift in the power of* 2 *stems from the fact that Atanassov's numbering starts with* $i = 1$*, whereas our numbering starts with* $i = 0$*).*

Let us return to the general case.

Theorem 4. *Let the infinite sequence* $\{c_i\}_{i\geq 0}$ *of arbitrary real (or complex) numbers be given, and let, for every* $i \geq 0$*,* $a_{i0} = a_{i(2i)} = c_i$*. Then the generating function for the elements of the middle column is given by*

$$\sum_{i\geq 0} a_{ii}t^i = \sum_{k\geq 0} \frac{q^{\binom{k+1}{2}} c_k t^k}{\prod_{i=0}^{k}(1 - q^i t)}.$$

Proof. By Proposition 1, we have that

$$\sum_{i\geq 0} a_{ii}t^i = \sum_{i\geq 0} \left(\sum_{k=0}^{i} q^{\binom{k+1}{2}} \binom{i}{k}_q c_k \right) t^i = \sum_{k\geq 0} q^{\binom{k+1}{2}} \left(\sum_{i\geq 0} \binom{i+k}{k}_q t^i \right) c_k.$$

By the fact that $\frac{1}{\prod_{k=0}^{n-1}(1 - q^k t)} = \sum_{k\geq 0} \binom{n-1+k}{k}_q t^k$, we obtain that

$$\sum_{i\geq 0} a_{ii}t^i = \sum_{i\geq 0} \left(\sum_{k=0}^{i} q^{\binom{k+1}{2}} \binom{i}{k}_q c_k \right) t^i = \sum_{k\geq 0} \frac{q^{\binom{k+1}{2}} c_k t^k}{\prod_{i=0}^{k}(1 - q^i t)},$$

as required. □

Above, we prescribed the elements on the left (and right) boundary diagonal and determined the remaining elements of the q-triangle using the defining recurrences, in particular the elements of the middle column. It is also possible to turn the perspective and prescribe the elements of the middle column. In the case $q = 1$, Atanassov [2, 3] studied this relation between these sequences of elements for several well-known sequences of numbers. By induction, we obtain the following q-analog of his result [2, Page 33].

Proposition 5. *Let the infinite sequence* $\{d_i\}_{i\geq 0}$ *of arbitrary real (or complex) numbers be given, and let, for every* $i \geq 0$*,* $a_{ii} = d_i$*. Then, for all* $0 \leq j \leq i - 1$*, one has that*

$$a_{ij} = a_{i(2i-j)} = q^{j-\binom{i+1}{2}} \sum_{k=0}^{i-j} (-1)^k q^{\binom{k}{2}} \binom{i-j}{k}_q d_{i-k}.$$

Next, we present some examples for the above proposition, where we use (1).

Example 6. *If one has in the middle column the consecutive q-numbers (for $q = 1$, see [2, Page 33]), that is, $d_i = [i+1]_q = \frac{1-q^{i+1}}{1-q}$, the elements on the left and right diagonal must be*

$$\begin{aligned} a_{i0} = a_{i(2i)} &= q^{-\binom{i+1}{2}} \sum_{k=0}^{i} (-1)^k q^{\binom{k}{2}} \binom{i}{k}_q [i+1-k]_q \\ &= q^{-\binom{i+1}{2}} \sum_{k=0}^{i} (-1)^k q^{\binom{k}{2}} \binom{i}{k}_q + q^{1-\binom{i+1}{2}} \sum_{k=0}^{i} (-1)^k q^{\binom{k}{2}} \binom{i}{k}_q [i-k]_q \\ &= q^{-\binom{i+1}{2}} \sum_{k=0}^{i} (-1)^k q^{\binom{k}{2}} \binom{i}{k}_q + [i]_q q^{1-\binom{i+1}{2}} \sum_{k=0}^{i-1} (-1)^k q^{\binom{k}{2}} \binom{i-1}{k}_q \\ &= q^{-\binom{i+1}{2}} \prod_{k=0}^{i-1} (1-q^k) + [i]_q q^{1-\binom{i+1}{2}} \prod_{k=0}^{i-2} (1-q^k), \end{aligned}$$

which implies that $a_{00} = 1$, $a_{10} = 1$, and $a_{i0} = 0$ for all $i \geq 2$. Thus, the sequence of elements $1, 1, 0, 0, 0, \ldots$ on the boundary diagonals is the same as in the case $q = 1$.

Example 7. *If one has in the middle column the consecutive squares of the q-numbers (for $q = 1$, see [2, Page 33]), that is, $d_i = [i+1]_q^2$, the elements on the left and right diagonal must be*

$$\begin{aligned} a_{i0} = a_{i(2i)} =& q^{-\binom{i+1}{2}} \sum_{k=0}^{i} (-1)^k q^{\binom{k}{2}} \binom{i}{k}_q [i+1-k]_q^2 \\ =& q^{-\binom{i+1}{2}} \sum_{k=0}^{i} (-1)^k q^{\binom{k}{2}} \binom{i}{k}_q + 2q^{1-\binom{i+1}{2}} \sum_{k=0}^{i} (-1)^k q^{\binom{k}{2}} \binom{i}{k}_q [i-k]_q \\ &+ q^{-\binom{i+1}{2}} \sum_{k=0}^{i} (-1)^k q^{\binom{k}{2}} \binom{i}{k}_q [i-k]_q^2, \end{aligned}$$

which, by the previous example, implies that

$$a_{i0} = a_{i(2i)} = q^{-\binom{i+1}{2}} \left(\prod_{k=0}^{i-1} (1-q^k) + (1+2q)[i]_q \prod_{k=0}^{i-2} (1-q^k) + [i]_q [i-1]_q \prod_{k=0}^{i-3} (1-q^k) \right),$$

that is, $a_{00} = 1$, $a_{10} = \frac{1+2q}{q}$, $a_{20} = \frac{1+q}{q^3}$ and $a_{i0} = 0$ for all $i \geq 3$. For $q \to 1$, the sequence of elements on the diagonals becomes $1, 3, 2, 0, 0, 0, \ldots$, as mentioned in [2].

Example 8. *If one has in the middle column the consecutive inverses of the q-numbers, that is, $d_i = [i+1]_q^{-1}$, the elements on the left and right diagonal must be*

$$\begin{aligned} a_{i0} = a_{i(2i)} &= q^{-\binom{i+1}{2}} \sum_{k=0}^{i} (-1)^k q^{\binom{k}{2}} \binom{i}{k}_q \frac{1}{[i+1-k]_q} \\ &= \frac{1}{q^{\binom{i+1}{2}} [i+1]_q} \sum_{k=0}^{i} (-q)^{-k} q^{\binom{k+1}{2}} \binom{i+1}{k+1}_q \\ &= \frac{(-1)^{i-1}}{[i+1]_q} \left(\prod_{k=0}^{i} (1-q^{k-1}) - 1 \right), \end{aligned}$$

that is, $a_{00} = 1$, $a_{i0} = \frac{(-1)^i}{[i+1]_q}$ for all $i \geq 1$.

In analogy to the above Theorem 4, it is possible to express the generating function for the elements on the boundary diagonals in terms of the elements d_i of the middle column. Since the elements on the right diagonal equal the elements on the left diagonal, it suffices to consider the former.

Theorem 9. *Let the infinite sequence $\{d_i\}_{i\geq 0}$ of arbitrary real (or complex) numbers be given, and let, for every $i \geq 0$, $a_{ii} = d_i$. Then the generating function for the elements on the left diagonal is given by*

$$\sum_{i\geq 0} a_{i0}t^i = \sum_{j\geq 0} \frac{q^{j+1}d_j t^j}{\prod_{k=1}^{j+1}(t+q^k)}.$$

Proof. By Proposition 5, we have that

$$\sum_{i\geq 0} a_{i0}t^i = \sum_{i\geq 0} \left(q^{-\binom{i+1}{2}} \sum_{k=0}^{i} (-1)^k q^{\binom{k}{2}} \binom{i}{k}_q d_{i-k} \right) t^i,$$

which is equivalent to

$$\sum_{i\geq 0} a_{i0}t^i = \sum_{j\geq 0}\sum_{i\geq 0} \left((-1)^i \binom{i+j}{i}_q \frac{t^i}{q^{(j+1)i}} \right) q^{\binom{j+1}{2}-(j+1)j} d_j t^j.$$

By the fact that $\frac{1}{\prod_{k=0}^{n-1}(1-q^k t)} = \sum_{k\geq 0} \binom{n-1+k}{k}_q t^k$, we obtain that

$$\sum_{i\geq 0} (-1)^i \binom{i+j}{i}_q \frac{t^i}{q^{(j+1)i}} = \prod_{k=1}^{j+1} \frac{1}{1+q^{-k}t} = q^{\binom{j+2}{2}} \prod_{k=1}^{j+1} \frac{1}{t+q^k}.$$

Thus,

$$\sum_{i\geq 0} a_{i0}t^i = \sum_{j\geq 0} \frac{q^{j+1}d_j t^j}{\prod_{k=1}^{j+1}(t+q^k)},$$

as required. □

Atanassov [3] considered as a particular sequence the Fibonacci numbers F_n. On the one hand, one may choose the sequence $c_i = F_i$ and prescribe them on the left and right diagonal, that is, $a_{i0} = a_{i(2i)} = F_i$, and determine the middle column. Atanassov mentioned without proof that $a_{ii} = F_{2i}$. In fact, using the second equation of Proposition 1, one finds for $q \to 1$ that

$$a_{ii} = \sum_{k=0}^{i} \binom{i}{k} F_k = F_{2i},$$

using in the second equation a well-known relation for the Fibonacci numbers (see, e.g., [1]). In a similar fashion, one can prescribe the Fibonacci numbers on the middle column, that is, $a_{ii} = F_i$ and determine the elements on the diagonal. Atanassov mentioned without proof that $a_{i0} = (-1)^{i+1}F_i$, for $i \geq 1$. Using Proposition 5 for $q \to 1$ with $d_i = F_i$, one finds that

$$a_{i0} = \sum_{k=0}^{i} (-1)^k \binom{i}{k} F_{i-k} = (-1)^{i+1}F_i,$$

using in the second equation another well-known relation for the Fibonacci numbers. Now, it seems to be interesting to consider a q-analog of this situation. A q-analog of the Fibonacci numbers satisfying $F_0(q) = 0$, $F_1(q) = 1$, and $F_{n+1}(q) = F_n(q) + q^{n-1}F_{n-1}(q)$ was considered by Schur [8] (see also [1, 5]). The first study would be to let $a_{i0} = a_{i(2i)} = F_i(q)$ and try to find a relation between a_{ii} and $F_{2i}(q)$, generalizing the relation of the undeformed case; conversely, the second study would be to let $a_{ii} = F_i(q)$ and try to find a_{i0}.

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