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Primes within generalized Fibonacci sequences

J. V. Leyendekkers[1] and A. G. Shannon[2,3]

[1] Faculty of Science, The University of Sydney
NSW 2006, Australia
[2] Faculty of Engineering & IT, University of Technology, Sydney
NSW 2007, Australia
[3] Campion College, PO Box 3052, Toongabbie East
NSW 2146, Australia
e-mails: t.shannon@ campion.edu.au,
Anthony.Shannon@uts.edu.au

Abstract: The structure of the 'Golden Ratio Family' is consistent enough to permit the primality tests developed for φ_5 to be applicable. Moreover, the factors of the composite numbers formed by a prime subscripted member of the sequence adhere to the same pattern as for φ_5. Only restricted modular class structures allow prime subscripted members of the sequence to be a sum of squares. Furthermore, other properties of φ_5 are found to apply to those other members with structural compatibility.
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1 Introduction

The analysis of the structure of the Golden Ratio family $\{\varphi_a\}$ can generate sets of generalized Fibonacci sequences [1, 9, 16]. Following on from a study of Fibonacci primes of the first member ($a = 5$) of this 'family' [3–8], obvious questions are:

- Do the generalized Fibonacci primes thus generated have properties similar to those of the ordinary Fibonacci sequence?
- Do the primality tests for the ordinary Fibonacci sequence apply more generally?

2 Calculations

We have previously [9] shown that the modular-ring structure is critical in the formation of the Golden Ratio family of generalized Fibonacci sequences, which are generated by

$$F_{\varphi_a}(n+1) = F_{\varphi_a}(n) + r_1 F_{\varphi_a}(n-1),\ n > 1, \tag{2.1}$$

where

$$\varphi_a = \frac{1+\sqrt{a}}{2} \tag{2.2}$$

and with initial conditions $F_{\varphi_a}(1) = F_{\varphi_a}(2) = 1$, and where r_1 is the row of $a \in \bar{1}_4 \subset Z_4$, the modular ring displayed in Table 1.

Row $r_i \downarrow$	**Class** $i \rightarrow$	$\bar{\mathbf{0}}_4$	$\bar{\mathbf{1}}_4$	$\bar{\mathbf{2}}_4$	$\bar{\mathbf{3}}_4$	**Comments**
0		0	1	2	3	$N = 4r_i + i$
1		4	5	6	7	even $\bar{0}_4$, $\bar{2}_4$
2		8	9	10	11	$(N^n, N^{2n}) \in \bar{0}_4$
3		12	13	14	15	odd $\bar{1}_4, \bar{3}_4; N^{2n} \in \bar{1}_4$

Table 1. Classes and rows for Z4

The class $\bar{1}_4 \subset Z_4$ has many unique features [11]. For example, for odd integers only those in this class can equal a sum of squares and even powers as in Table 1. Because of the power restriction in regions where even powers are plentiful there is 'more room' in class $\bar{3}_4$ for primes, an anomaly noted in [14].

All $\varphi_a(p) \in \bar{1}_4$ (p prime) could be a sum of squares. When a = 5, r_1 = 1 for a sum of squares in the well-known form [11, 12]

$$F_p = F^2_{\frac{p+1}{2}} + F^2_{\frac{p-1}{2}} \tag{2.3}$$

In Section 4, we shall consider other Fibonacci sequences with this. We note that when r_1 is odd the parity structure of the sequences conforms to the pattern odd – odd – even – odd – odd – even …, but when r_1 is even we get odd only in the sequence (Table 2), a fact which follows from Table 1 and Equation (2.1).

a	$\boldsymbol{r_1}$	**Class of** $\boldsymbol{r_1}$	**Parity structure of** $\boldsymbol{\varphi_a(n)}$	37	9	$\bar{1}_4$	ooe
5	1	$\bar{1}_4$	ooe	41	10	$\bar{2}_4$	ooo
13	3	$\bar{3}_4$	ooe	53	13	$\bar{1}_4$	ooe
17	4	$\bar{0}_4$	ooo	57	14	$\bar{2}_4$	ooo
29	7	$\bar{3}_4$	ooe	61	15	$\bar{3}_4$	ooe

Table 2. Parity structure of $\varphi_a(n)$ [o: odd; e: even]

When an element of any sequence is at a point where $n = p$ (prime), the class of r_1 determines the class of $\varphi_a(p)$ as in Table 3 where it is clear that the row of $a \in \bar{1}_4$ dominates the characteristics of each sequence.

Class of r_1	**Class of $\varphi_a(p)$**	***a***
$\bar{0}_4, \bar{1}_4$	all $\bar{1}_4$	5, 17, 37, 53, ...
$\bar{3}_4$	$\bar{1}_4\bar{3}_4\bar{1}_4\bar{3}_4$...	13, 29, 61
$\bar{2}_4$	all $\bar{3}_4$	41, 57, ...

Table 3. Class of $\varphi_a(p)$

For the first member of the Golden Ratio family, $a = 5$, and when n is a prime p, $F_{\varphi_a}(p)$ can also be prime [17]. When $n \neq p$, no prime is formed except for $p = 3$. This seems to apply for all members of the Golden Ratio family of Fibonacci sequences. An interesting structural feature is that when $p = a, \varphi_a(p)$ is always composite with the smallest factor equal to a (Table 4).

a* = *p	$F_{\varphi_a}(p)$	**Factors**
5	5	5×1
13	14209	13×1093
17	2135149	17×125597
29	77433768659591	29×2670129953779
37	342997157992347255561	$37 \times 9270193459252628530$

Table 4. Factors when $p = a$

3 Examples of primes in sequences for *a* = 13, 17

The prime and composite integers generated for $a = 13$ (r_1 odd) and $a = 17$ (r_1 even) are set out in Tables 5 and 6 where

$$\text{Factors of } N_p = kp \pm 1 \qquad (3.1)$$

It is found that the same criteria for distinguishing primes and composites when $a = 5$ [3–8] also apply for higher members of the Golden Ratio family.

For $a = 5$, k is often equal to 2 [5], and this is the case for $F_{\varphi_{13}}(19)$ in Table 5. These results suggest that (3.1) is general for the family.

p	$F_{\varphi_{13}}(p)$	c/p	k	Factors	$F_{\varphi_{17}}(p)$	c/p	k	Factors
3	4	c	2	2×2	5	p	–	–
5	19	p	–	–	29	p	–	–
7	97	p	–	–	181	p	–	–
11	2683	p	–	–	7589	p	–	–
13	14209	c	84(+)	13 1093 = $84p + 1$	49661	c	4(+1) 72(+1)	53 = $4p + 1$ 937 = $72p + 1$
17	399331	c	6(+1) 228(+1)	103 = $6p + 1$ 3877 = $228p + 1$	2135149	c	7388(+1)	17 125597 = $7388p + 1$
19	2117473	c	2(–) 8(–) 20(–)	37 = $2p – 1$ 151 = $8p – 1$ 379 = $20p – 1$	14007941	c	2(–1) 19926(–1)	37 = $2p–1$ 378593 = $19926p – 1$

Table 5. Factors of $N_p = kp \pm 1$ ($a = 13$)   Table 6: Factors of $N_p = kp \pm 1$ ($a = 17$)

4 Sum of squares

The structure of the various sequences of the Golden Ratio family suggests that only $r_1 \in \{\bar{0}_4, \bar{1}_4\}$ will have all $\varphi_a(p) \in \bar{1}_4$ (Table 3) and these can equal a sum of squares; that is, $\varphi_a(p) = x^2 + y^2$. The (x, y) couples for $\varphi_5(p)$ may be calculated from Equation (2.1) whereas, in general, the $\varphi_a(p)$ (x, y) couples may be calculated from (x odd, y even) [10]:

$$x, y = \tfrac{1}{2}\left(A \pm \sqrt{2F_{\varphi_a}(p) - A^2}\right), \; A = x + y. \tag{4.1}$$

The upper limit for A is $\sqrt{2F_{\varphi_a}(p)}$ and when $F_{\varphi_a}(p)$ is prime it is close to $\sqrt{2F_{\varphi_a}(p)}$. Moreover, A^*, the right-end-digit (RED) of A [10, 13] is restricted so that (x, y) estimates are easier to find (Table 7).

$\varphi_a(p)^*$	A^*
1	1, 9
3	1, 5, 9
7	3, 5, 7
9	3, 7

Table 7. REDs for A

Composite numbers either have one (x, y) couple with common factors or the same number as the number of factors, but primes have only one (x, y) couple with no common factors (Table 8).

p	$\varphi_{17}(p)$	Status	x, y	Factors
3	5	p	1, 2	–
5	29	p	5, 2	–
7	181	p	9, 10	–
11	7589	p	65, 58	–
13	49661	c	181, 130 85, 206	53×937
17	2135149	c	1165, 882 1443, 230	17×125597
19	14007941	c	2929, 2330 2015, 3154	37×378593

Table 8. Numbers of factors associated with $\varphi_{17}(p)$

It is significant that the odd and even (*x*, *y*) in Table 8 are elements of the classes $\bar{1}_4$, $\bar{2}_4$, respectively, but for those $\varphi_a(p)$ with $r_1 \in \bar{3}_4$ (for example, *a* = 13, 29, 61) there are *x*, *y* couples alternatively, that is, when $\varphi_a(p) \in \bar{1}_4$. However, if the factors are in class $\bar{3}_4$, then *x*, *y* couples do not form. When there is an odd number of factors some must be in class $\bar{1}_4$ since $\bar{3}_4\bar{3}_4 \in \bar{1}_4$ but $\bar{3}_4\bar{3}_4\bar{3}_4 \in \bar{3}_4$. Thus, for $\varphi_{17}(p)$, $x = \varphi_{17}\left(\frac{1}{2}(p+1)\right)$, $y = 2\varphi_{17}\left(\frac{1}{2}(p-1)\right)$. For instance, for $\varphi_{17}(13)$, $x = \varphi_{17}(7)$, $y = 2\varphi_{17}(6)$ which should be compared with (2.3).

5 Sequences structurally compatible with φ_5

These have the same parity structure and r_1 of *a* is an element of class $\bar{1}_4$.

(a) The first of these is $\varphi_{37}(p)$ (Table 9). The factors are in class $\bar{1}_4$. They conform with (3.1) and the *x*, *y* couples are $x = \varphi_{37}\left(\frac{1}{2}(p+1)\right)$, $y = 3\varphi_{37}\left(\frac{1}{2}(p-1)\right)$ rather than (2.3).

p	$\varphi_{37}(p)$	Factors	Class of factors	k (3.1)	x, y
5	109(p)	–	–	–	3, 10
7	1261(c)	13×97	$\bar{1}_4\,\bar{1}_4$	13 = 2*p* – 1 97 = 14*p* – 1	19, 30 35, 6
11	185329(c)	241×769	$\bar{1}_4\,\bar{1}_4$	241 = 22*p* – 1 769 = 70*p* – 1	327, 280 423, 80
13	2295721(p)	–	–	–	1261, 840

Table 9. *x*, *y* couples for $\varphi_{37}(p)$

(b) The second of these is $\varphi_{53}(p)$ (Table 10). For the smallest p values the $\varphi_{53}(p)$ integers are composite, and the factors do not always fall in class $\bar{1}_4$. Moreover, only a modified form of Equation (2.3) holds. Clearly, changes in the structure lead to more complex systems.

p	$\varphi_{53}(p)$	Factors	Class of factors	k (3.1)	x, y
5	209	11×19	$\bar{3}_4\ \bar{3}_4$	$\begin{cases}11=2p+1\\19=4p-1\end{cases}$	–
7	3277	29×113	$\bar{1}_4\ \bar{1}_4$	$\begin{cases}29=4p+1\\113=16p+1\end{cases}$	26, 51 19, 54
11	881453	331×2663	$\bar{3}_4\ \bar{3}_4$	$\begin{cases}331=30p+1\\2663=242p+1\end{cases}$	–

Table 10. x, y couples for $\varphi_{53}(p)$

6 *a* as a composite element of $\bar{1}_4$

Here the same structural characteristics occur as with primes in Section 2. The rows of $\varphi_9, \varphi_{25} \in \bar{2}_4$ so that $\varphi_9(p), \varphi_{25}(p) \in \bar{3}_4$ and cannot form a sum of squares (Tables 8, 9). While φ_9 produces many primes, the higher a values only contain a scatter of primes (Table 11). The factors of prime positioned numbers satisfy Equation (3.1).

p	$\varphi_9(p)\in\bar{3}_4$	$\varphi_{21}(p)\in\bar{1}_4$	$\varphi_{25}(p)\in\bar{3}_4$	$\varphi_{33}(p)\in\bar{1}_4$
r_1	**2**	**5**	**6**	**8**
3	3(p)	$6 = 2\times 3$	7(p)	$9 = 3\times 3$
5	11(p)	41(p)	$55 = 5\times 11$ $5, 11 = 2p + 1$	89(p)
7	43(p)	$301 = 7\times 43$ $7, 43 = 6p + 1$	463(p)	937(p)
11	683(p)	$17621 = 67\times 263$ $67 = 6p + 1$ $263 = 24p - 1$	35839(p)	$113993 = 11\times 43\times 241$ $11, 43 = 4p - 1$ $241 = 22p - 1$
13	2731(p)	$136681 = 103\times 1327$ $103 = 8p - 1$ $1327 = 102p + 1$	$32053 = 79\times 4057$ $79 = 6p + 1$ $4057 = 312p + 1$	$1282969 = 261\times 491559$ $261 = 20p + 1$ $491559 = 37812p + 3$
17	48691(p)	8275601(p)	25854247(p)	164643641(p)
19	174763(p)	164457461(p)	$232557151 = 419\times 555029$ $419 = 22p + 1$ $555029 = 29212p + 1$	1869986953(p)

Table 11. Some factors of prime positioned numbers [r_1 is a row of a]

6 Final comments

The closer the structures of the members of the golden ratio family are to one another the more similarities there will be. For example, when $\varphi_a \in \bar{1}_4$ and $r_1 \in \{\bar{0}_4, \bar{1}_4\}$ the more similarities there will be, particularly for the primality tests developed for φ_5 [3–8]. The classes of $\varphi_a(n)$ and r_1 also classify the sums of squares (Tables 2, 3, 8), and the structure is paramount for analysis of the infinity of sequences arising from Equations (2.1) and (2.2).

Another feature of φ_5 found to be important [5, 12] is that the triple (F_{p+1}, F_p, F_{p-1}) has either F_{p+1} or F_{p-1} divisible by p. Furthermore, the other two members of the triple ±1 are also divisible by p, but if $p = a$, then F_p is divisible by p (Tables 4 and 12). This divisibility feature permits a check on the accuracy of elements of the sequence when approximating them by the power of the relevant golden ratio or to reduce the triples to their primitive forms [3].

- For $a = 5, \varphi_5$, $p|F_{p+1}$ or $p|F_{p-1}$ can be predicted, but when $a > 5$ this prediction is more complex, though commonly $p|F_{p+1}$ and when a is composite the divisibility is more consistent.
- For instance, for φ_9, only $p \mid \varphi_9(p-1)$ occurs, while for φ_{33}, $p \mid \varphi_{33}(p+1)$ is preferred.
- For $\varphi_{29}(7)$ none of the triples is divisible by 7, and
- for $\varphi_{53}(13)$ none is divisible by 13.

Apparently when $r_1 = p$ (for φ_{29}, $r_1 = 7$ and for φ_{53}, $r_1 = 13$) none of the triples is divisible by the prime if it equals r_1.

p'	$\varphi_{13}(p')$	$\varphi_{17}(p')$	$\varphi_{29}(p')$	$\varphi_{37}(p')$
5	+	+	–	+
7	+	+	–	–
11	+	+	+	–
13	p	–	–	+
17	–	p	+	+
19	+	–	+	+
23	–	+	–	+
29	–	+	p	+
31	+	+	+	+

Table 12. p'; $p'|F_p$ where +: $p'|\varphi_p(p'+1)$; –: $p'|\varphi_p(p'-1)$

The interested reader may enjoy the challenge of extending these results to examine Havil's claim that the Golden Ratio is the world's most irrational number and the first of what he calls 'awkward' numbers [2] and whether there are similar families of meta-Fibonacci sequences such as [15]

$$F_n = F_{n-F_{n-1}} + F_{n-1-F_{n-2}} \tag{7.1}$$

with the usual initial conditions.

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