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On Diophantine triples and quadruples

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Abstract: In this paper we consider Diophantine triples $\{a, b, c\}$, (denoted $D(n)$-3-tuples) and give necessary and sufficient conditions for existence of integer n given the 3-tuple $\{a, b, c\}$, so that $ab + n,\ ac + n,\ bc + n$ are all squares of integers. Several examples as applications of the main results, related to both Diophantine triples and quadruples, are given.
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1 Introduction

Definition 1.1. *A set of m positive integers is called a Diophantine m-tuple with the property $D(n)$ or simply $D(n)$-m-tuple, if the product of any two elements of this set increased by n is a perfect square.*

As a special case, a Diophantine m-tuple is a set of m positive integers with the property: the product of any two of them increased by one unit is a perfect square, for example, {1, 3, 8, 120} is a Diophantine quadruple, since we have

$$1 \times 3 + 1 = 2^2, 1 \times 8 + 1 = 3^2, 1 \times 120 + 1 = 11^2,$$

$$3 \times 8 + 1 = 5^2, 3 \times 120 + 1 = 19^2, 8 \times 120 + 1 = 31^2.$$

The study of Diophantine m-tuple can be traced back to the third century AD, when the Greek mathematician Diophantus discovered that $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$ is a set of four rationals which has the above property. Then Fermat obtained the first Diophantine quadruple $\{1,\ 3,\ 8,\ 120\}$. Astoundingly, $\frac{777480}{8288641}$ was found to extend the Fermat's set to $\left\{1,3,8,120,\frac{777480}{8288641}\right\}$ and then any two elements of this set increased by one unit is a perfect square of a rational number, which was Euler's contribution. Moreover, he acquired the infinite family of Diophantine quadruple $\{a,b,a+b+2r,4r\left(r+a\right)\left(r+b\right)\}$, if $ab+1=r^2$. In January 1999, Gibbs [7] found the first set of six positive rationals with the above property.

In the integer case, there is a famous conjecture: there does not exist a Diophantine quintuple.

The case $n\neq 1$ also have been studied by several mathematicians, for example, $\{1,2,5\}$ is a $D(-1)$-triple. It is interesting to note that if n is an integer of form $n=4k+2$, then there does not exist a Diophantine quadruple with the property $D(n)$. This theorem has been independently proved by Brown [1], Gupta and Singh [8] and Mohanty and Ramasamy [11] all in 1985. In 1993, Dujella [2] proved that if an integer n does not have the form $n=4k+2$ and $n\notin S=\{-4,-3,-1,3,5,8,12,20\}$, then there exists at least one Diophantine quadruple with the property $D(n)$.

In the case $n=-1$, the conjecture—there does not exist a $D(-1)$-quadruple is known as $D(-1)$-quadruple conjecture. In 1985, Brown'[1] proved the nonextendability of the Diophantine $D(-1)$ triple $\{1,2,5\}$. Walsh [13] and Kihel [9] also independently proved that in 1999 and 2000 respectively. In 1984, Mohanty and Ramasamy [10] proved that the Diophantine $D(-1)$ triple $\{1,5,10\}$ can not be extended to a $D\left(-1\right)$ quadruple. More recently in [14] was another proof of nonextendibility of $\{1,2,5\}$ and $\{1,5,10\}$ using properties of Lucas and Fibonacci numbers.

Furthermore, Brown [1] proved that $\left\{n^2+1,\left(n+1\right)^2+1,\left(2n+1\right)^2+4\right\}$ can not be extended to a Diophantine quadruple with the property $D\left(-1\right)$ if $n\equiv 0\,(\mathrm{mod}4)$. Dujella [3] was the first mathematician who proved the nonextendability for all triples of the form $\{1,2,c\}$ in 1998. The endeavor in proving that $\{1,5,c\}$ can not be extended was mostly attributed to Muriefah and Al-Rashed [12]. In 2005, Filipin [6] proved the nonextendability of $\{1,10,c\}$.

There does not exist a Diophantine quintuple with the property $D(-1)$. This was proved by Dujella and Fuchs [5] in 2005. Moreover, in 2007, Dujella, Filipin and Fuchs [4] proved that there are only exist finitely many quadruples with the property $D(-1)$.

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Many of the current results are from first authors master's thesis [15], 2011.

2 Necessary and sufficient conditions

Definition 2.1. *For three given positive integers a, b, c with $a<b<c$, $\{a,b,c\}$ is called a Diophantine triple with the property $D\left(n\right)$ if there exist positive integers α, β, γ and an integer n satisfying*

$$\begin{cases}\alpha^2=ab+n,\\ \beta^2=ac+n,\\ \gamma^2=bc+n.\end{cases}\tag{2.1}$$

Example 2.2. *Given* $\{a,b,c\} = \{12,28,42\}$*, then* $(\alpha,\beta,\gamma) = (1,13,29)$ *with* $n = -335$ *and* (α,β,γ)=$(11,17,31)$ *with* $n = -215$ *satisfying:*

$$\begin{cases} \alpha^2 = 1^2 = 12\times 28 - 335 = ab+n, \\ \beta^2 = 13^2 = 12\times 42 - 335 = ac+n, \\ \gamma^2 = 29^2 = 28\times 42 - 335 = bc+n. \end{cases} \quad \begin{cases} \alpha^2 = 11^2 = 12\times 28 - 215 = ab+n, \\ \beta^2 = 17^2 = 12\times 42 - 215 = ac+n, \\ \gamma^2 = 31^2 = 28\times 42 - 215 = bc+n. \end{cases}$$

Remark 2.3. *For three given positive integers* α, β, γ *with* $\alpha < \beta < \gamma$*, there can exist more than one Diophantine triple satisfying* (2.1) *with the property* $D(n)$ *(i.e. have different* $n's$*).*

Example 2.4. *Given* $(\alpha,\beta,\gamma) = (3,11,31)$*, then exist Diophantine triples* $\{1,8,120\}$ *with* $n = 1$, $\{8,28,42\}$ *with* $n = -215$ *and* $\{14,34,42\}$ *with* $n = -467$ *satisfying* (2.1).

Theorem 2.5. *If* $a+b+c$ *is even with* $a < b < c$*, then* $\{a,b,c\}$ *is a Diophantine triple with the property* $D(n)$ *where*

$$\begin{cases} \alpha = \frac{1}{2}(a+b-c) \\ \beta=\frac{1}{2}(c+a-b) \\ \gamma = \frac{1}{2}(b+c-a) \end{cases} \text{and } n = \frac{1}{4}\left(a^2+b^2+c^2-2ab-2ac-2bc\right).$$

Proof. Since $a+b+c$ is even, $a+b-c$, $c+a-b$ and $b+c-a$ are all even. Then α, β and γ are all integers.

Let $n = \frac{1}{4}\left(a^2+b^2+c^2-2ab-2ac-2bc\right)$,
then $ab+n = ab+\frac{1}{4}\left(a^2+b^2+c^2-2ab-2ac-2bc\right) = \left[\frac{1}{2}(a+b-c)\right]^2 = \alpha^2$.

Similarly, we can get $ac+n = \beta^2$ and $bc+n = \gamma^2$. The results for β and γ follow easily.

Then the triple $\{a,b,c\}$ is a Diophantine triple with the property $D(n)$. □

Remark 2.6. *For any given three positive integers* α,β,γ *with* $\alpha < \beta < \gamma$*, there exist at lease two Diophantine triples* $\{a,b,c\}$ *that satisfy* (2.1)*. The general solutions are given by* $\begin{cases} a = \alpha+\beta \\ b = \alpha+\gamma \\ c = \beta+\gamma \end{cases}$,

$n = -(\alpha\beta+\alpha\gamma+\beta\gamma)$ *and* $\begin{cases} a = \beta-\alpha \\ b = \gamma-\alpha \\ c = \beta+\gamma. \end{cases}$ $n = \alpha\beta+\alpha\gamma-\beta\gamma$

Proof. These can easily be proved by checking the equality of $ab+n$ and α^2, $ac+n$ and β^2, $bc+n$ and γ^2.

For example, $ab+n = (\alpha+\beta)(\alpha+\gamma)+n = \alpha^2+\alpha\gamma+\alpha\beta+\beta\gamma-(\alpha\gamma+\alpha\beta+\beta\gamma) = \alpha^2$. □

These general solutions could be explained by matrix equations as well:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \alpha+\beta \\ \alpha+\gamma \\ \beta+\gamma \end{bmatrix},$$

or

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \beta - \alpha \\ \gamma - \alpha \\ \beta + \gamma \end{bmatrix},$$

and

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{2} \begin{bmatrix} a+b-c \\ a-b+c \\ -a+b+c \end{bmatrix}.$$

The main result is

Theorem 2.7. *Let a, b, c be three positive integers with $a < b < c$, such that a, b, c are all odd or all even or only one of them is odd. Then exists an integer n such that $\{a, b, c\}$ is a Diophantine triple with the property $D(n)$ if and only if there exists positive integers λ, P_b, P_c such that $\lambda = \frac{b(c-a)}{4P_b} + P_b = \frac{c(b-a)}{4P_c} + P_c$. Moreover, $n = \lambda^2 - bc$.*

Proof. ($\Rightarrow$) Suppose that a, b, c are three positive integers with $a < b < c$, the number of even integers in $\{a, b, c\}$ is not one, and $\{a, b, c\}$ is a Diophantine triple. By assumption there exist integers α, β, γ and an integer n such that $\begin{cases} \alpha^2 = ab + n, \\ \beta^2 = ac + n, \\ \gamma^2 = bc + n. \end{cases}$

Now, denote $P_b = \frac{\gamma-\alpha}{2}$, $P_c = \frac{\gamma-\beta}{2}$. Subtracting the first equation from the third one, we get $b(c-a) = \gamma^2 - \alpha^2$.

The number of even integers in $\{a, b, c\}$ is not one,
$\Rightarrow 2|b(c-a) = \gamma^2 - \alpha^2 = (\gamma+\alpha)(\gamma-\alpha)$
$\Rightarrow 2|\gamma-\alpha$ or $2|\gamma+\alpha$
$\Rightarrow 2|\gamma-\alpha$ and $2|\gamma+\alpha$
$\Rightarrow P_b = \frac{\gamma-\alpha}{2}$ is an integer and $\frac{b(c-a)}{4P_b} = \frac{\gamma^2-\alpha^2}{4\frac{\gamma-\alpha}{2}} = \frac{\gamma+\alpha}{2}$ is an integer
$\Rightarrow \lambda = \frac{b(c-a)}{4P_b} + P_b$ is an integer.

Similarly, P_c and $\frac{c(b-a)}{4P_c}$ are integers by same argument.

Since $\frac{b(c-a)}{4P_b} + P_b = \frac{\gamma^2-\alpha^2}{4\frac{\gamma-\alpha}{2}} + \frac{\gamma-\alpha}{2} = \frac{\gamma+\alpha}{2} + \frac{\gamma-\alpha}{2} = \gamma$ and $\frac{c(b-a)}{4P_c} + P_c = \frac{\gamma^2-\beta^2}{4\frac{\gamma-\beta}{2}} + \frac{\gamma-\beta}{2} = \frac{\gamma+\beta}{2} + \frac{\gamma-\beta}{2} = \gamma$, $\frac{b(c-a)}{4P_b} + P_b = \frac{c(b-a)}{4P_c} + P_c$.

($\Leftarrow$) Suppose that a, b, c are three positive integers with $a < b < c$, the number of even integers in $\{a, b, c\}$ is not one and there exist positive integers λ, P_b, P_c such that $\lambda = \frac{b(c-a)}{4P_b} + P_b = \frac{c(b-a)}{4P_c} + P_c$.

Let $\gamma = \lambda$, $\alpha = \left|\frac{b(c-a)}{4P_b} - P_b\right|$ and $\beta = \left|\frac{c(b-a)}{4P_c} - P_c\right|$, $n = \lambda^2 - bc$, then
$ab + n = \lambda^2 + ab - bc = \left(\frac{b(c-a)}{4P_b} + P_b\right)^2 - b(c-a) = \left(\frac{b(c-a)}{4P_b}\right)^2 + \frac{b(c-a)}{2} + {P_b}^2 - b(c-a) = \left(\frac{b(c-a)}{4P_b}\right)^2 - \frac{b(c-a)}{2} + {P_b}^2 = \left(\frac{b(c-a)}{4P_b} - P_b\right)^2 = \alpha^2,$

$ac+n=\lambda^2+ac-bc=\left(\frac{c(b-a)}{4P_c}+P_c\right)^2-c\,(b-a)=\left(\frac{c(b-a)}{4P_c}\right)^2+\frac{c(b-a)}{2}+{P_c}^2-c\,(b-a)=\left(\frac{c(b-a)}{4P_c}\right)^2-\frac{c(b-a)}{2}+{P_c}^2=\left(\frac{c(b-a)}{4P_c}-P_c\right)^2=\beta^2$,
and, by above, $bc+n=\lambda^2=\gamma^2$. □

Remark 2.8. *If a, b, c are three positive integers with $a<b<c$ and only one of them is even, then $\{a,b,c\}$ is a Diophantine triple if and only if there exist positive integers λ, P_b, P_c such that $\lambda=\frac{b(c-a)}{P_b}+P_b=\frac{c(b-a)}{P_c}+P_c$. Moreover, $n=\frac{\lambda^2}{4}-bc$.*

Proof. ($\Rightarrow$) Denote $P_b=\gamma-\alpha$ and $P_c=\gamma-\beta$.

It is clear that P_b and P_c are positive integers. Since $\frac{b(c-a)}{P_b}+P_b=\frac{\gamma^2-\alpha^2}{\gamma-\alpha}+\gamma-\alpha=\gamma+\alpha+\gamma-\alpha=2\gamma$ and $\frac{c(b-a)}{P_c}+P_c=\frac{\gamma^2-\beta^2}{\gamma-\beta}+\gamma-\beta=\gamma+\beta+\gamma-\beta=2\gamma$, $\frac{b(c-a)}{P_b}+P_b=\frac{c(b-a)}{P_c}+P_c$.

($\Leftarrow$) Since a, b, c are three positive integers with $a<b<c$ and only one of them is even, then at least one of $b\,(c-a)$ and $c\,(b-a)$ is odd. Without loss of generality, we assume that $c\,(b-a)$ is odd, then P_c and $\frac{c(b-a)}{P_c}$ are odd. Thus λ is even, then $\frac{b(c-a)}{P_b}$ and P_b are both odd or even.

Let $\gamma=\frac{\lambda}{2}$, $\alpha=\frac{1}{2}\left|\frac{b(c-a)}{P_b}-P_b\right|$, $\beta=\frac{1}{2}\left|\frac{c(b-a)}{P_c}-P_c\right|$ and $n=\frac{\lambda^2}{4}-bc$, therefore α, β and γ are all positive integers. Then we have
$ab+n=\frac{\lambda^2}{4}-bc+ab=\frac{1}{4}\left(\frac{b(c-a)}{P_b}+P_b\right)^2-b\,(c-a)=\frac{1}{4}\left(\frac{b(c-a)}{P_b}\right)^2+\frac{b(c-a)}{2}+\frac{{P_b}^2}{4}-b\,(c-a)=\frac{1}{4}\left(\frac{b(c-a)}{P_b}\right)^2-\frac{b(c-a)}{2}+\frac{{P_b}^2}{4}=\frac{1}{4}\left(\frac{b(c-a)}{P_b}-P_b\right)^2=\alpha^2$,
$ac+n=\frac{\lambda^2}{4}-bc+ac=\frac{1}{4}\left(\frac{c(b-a)}{P_c}+P_c\right)^2-c\,(b-a)=\frac{1}{4}\left(\frac{c(b-a)}{P_c}\right)^2+\frac{c(b-a)}{2}+\frac{{P_c}^2}{4}-c\,(b-a)=\frac{1}{4}\left(\frac{c(b-a)}{P_c}\right)^2-\frac{c(b-a)}{2}+\frac{{P_c}^2}{4}=\frac{1}{4}\left(\frac{c(b-a)}{P_c}-P_c\right)^2=\beta^2$,
and, by above, $bc+n=\frac{\lambda^2}{4}=\gamma^2$. □

Corollary 2.9. *If the Diophantine triple $\{a,b,c\}$ is $D(n)$ for some set of integers $\{n\}$ then all values of n can be found from following Diophantine equation,*

$$xy(x-y)=Ax-By, \tag{2.2}$$

where $A=c(b-a)$, $B=b(c-a)$ if only one of a,b,c is even and $A=c(b-a)/4$, $B=b(c-a)/4$ otherwise.

Proof. Take $P_b=x, P_c=y$ in Remark 2.8, Theorem 2.7 and multiply by xy, then simplify. It is clear that any pair (x,y) that satisfies (2.2) yields a possible value of n. Conversely, any 3-tuple $\{a,b,c\}$ that is $D(n)$ must yield solution (x,y) satisfies (2.2), by Theorem 2.7, Remark 2.8. □

Remark 2.10. *It is possible Corollary 2.9 has further consequences, especially for 4-tuples, then this can be focus of some more works.*

3 Several applications

The $D(-1)$ triple $\{17, 26, 85\}$ is a simple starting example, see Remark 2.8. Take $P_b = c - a = 68$, $P_c = b - a = 85$, then $n = 94^2/4 - 26 \times 85 = -1$. The 'famous' triple $\{1, 8, 120\}$ is both $D(1)$, $D(721)$, from Theorem 2.7.

Lemma 3.1. *If a, b, c are three odd positive integers with $a < b < c$ and $\{a, b, c\}$ is a Diophantine triple with the property $D(n)$, then $4|(c-b)$, $4|(b-a)$ and $4|(c-a)$.*

Proof. By Definition 2.1, there exist positive integers α, β, γ and integer n so that

$$\begin{cases} \alpha^2 = ab + n, \\ \beta^2 = ac + n, \\ \gamma^2 = bc + n. \end{cases}$$

Then
$2|(c-b) \Rightarrow 2|a(c-b) = (ac+n) - (ab+n) = \beta^2 - \alpha^2 = (\beta - \alpha)(\beta + \alpha)$
$\Rightarrow 2|\beta - \alpha$ or $2|\beta + \alpha$
$\Rightarrow 2|\beta - \alpha$ and $2|\beta + \alpha$
$\Rightarrow 4|(\beta - \alpha)(\beta + \alpha) = \beta^2 - \alpha^2 = a(c-b)$
$\Rightarrow 4|(c-b)$.
Similarly, $4|(b-a)$ and $4|(c-a)$. $\square$

Corollary 3.2. *If $c > 7$ is a prime number, then $\{3, 7, c\}$ is never a Diophantine triple with the property $D(n)$ for any integer n.*

Proof. Suppose that $\{3, 7, c\}$ is a Diophantine $D(n)$ triple, then by Theorem 2.7, there exists positive integers λ, P_b, P_c such that

$$\lambda = \frac{7(c-3)}{4P_b} + P_b = \frac{4c}{4P_c} + P_c = \frac{c}{P_c} + P_c$$

by Theorem 2.7.

Then $\frac{c}{P_c} = \lambda - P_c$ is an integer, thus $P_c|c$.

Consider that c is a prime number, then $P_c = 1$ or $P_c = c$ and $\lambda = c + 1$.

According to Corollary 3.1, suppose that $c = 3 + 4k$ with integer $k > 1$, then we have $\lambda = \frac{7k}{P_b} + P_b = c + 1 = 3 + 4k + 1 = 4(k+1)$.

Let $A = \frac{7k}{P_b}$, $B = P_b$, we have $A + B = 4(k+1)$ and $AB = 7k$. So
$(A-B)^2 = (A+B)^2 - 4AB = (4(k+1))^2 - 28k$
$= 16k^2 + 32k + 16 - 28k = 4(4k^2 + k + 4)$.

Thus $A - B$ is even and $\frac{A-B}{2}$ is an integer, we have $\left(\frac{A-B}{2}\right)^2 = 4k^2 + k + 4$.

Since $(2k+1)^2 - (4k^2 + k + 4) = (4k^2 + 4k + 1) - (4k^2 + k + 4)$ $= 3(k-1) > 0$, we obtain that $(2k)^2 = 4k^2 < \left(\frac{A-B}{2}\right)^2 = 4k^2 + k + 4 < 4k^2 + 4k + 1 = (2k+1)^2$, which contradicts the fact that $\frac{A-B}{2}$ is an integer.

Thus if $c > 7$ is a prime number, then $\{3, 7, c\}$ is never a Diophantine triple with the property $D(n)$ for any integer n. $\square$

Remark 3.3. *If one of c and d is a prime number larger than 7, then for any integer n, the quadruple $\{3, 7, c, d\}$ is never a Diophantine quadruple with the property $D(n)$.*

Proof. This could be proved by directly applying Corollary 3.2. □

Corollary 3.4. *If a, b, c are three odd positive integers with $a < b < c$, $\frac{c-a}{4}$ and $\frac{b-a}{8}$ are odd integers, then for any positive integer n, the triple $\{a, b, c\}$ will never be a Diophantine triple with the property $D(n)$.*

Proof. Suppose that exists positive integer n such that $\{a, b, c\}$ is a Diophantine triple with the property $D(n)$. Then by Theorem 2.7, there exist positive integers λ, P_b, P_c such that $\lambda = \frac{b(c-a)}{4P_b} + P_b = \frac{c(b-a)}{4P_c} + P_c$.

Since $\frac{b(c-a)}{4P_b} \cdot P_b = \frac{b(c-a)}{4}$ is odd, $\frac{b(c-a)}{4P_b}$ and P_b are both odd integers, then $\lambda = \frac{b(c-a)}{4P_b} + P_b$ is even.

On the other hand, $\frac{b-a}{8}$ is odd implies that $\frac{b-a}{4}$ is even and $4 \nmid \frac{b-a}{4}$.

Then $2 | \frac{c(b-a)}{4P_c} \cdot P_c$ and $4 \nmid \frac{c(b-a)}{4P_c} \cdot P_c$.

Then $2 | \frac{c(b-a)}{4P_c}$ and $2 \nmid P_c$ or $2 \nmid \frac{c(b-a)}{4P_c}$ and $2 | P_c$, then $\lambda = \frac{c(b-a)}{4P_c} + P_c$ is odd, which contradicts that λ is even. □

Remark 3.5. *If a, b, c are three odd positive integers with $a < b < c$, $\frac{c-a}{8}$ and $\frac{b-a}{4}$ are odd integers, then for any positive integer n, the triple $\{a, b, c\}$ will never be a Diophantine triple with the property $D(n)$.*

Proof. The proof will be the same as Corollary 3.4. □

Corollary 3.6. *If a is an odd positive integer, then for any positive integer k_1, k_2, the triple $\{a, a + 8 + 16k_1, a + 4 + 8k_2\}$ is never a Diophantine triple with the property $D(n)$ for any integer n.*

Proof. Since $\frac{1}{4}(a + 4 + 8k_2 - a) = 1 + 2k_2$ and $\frac{1}{8}(a + 8 + 16k_1 - a) = 1 + 2k_1$ are both odd, by Corollary 3.4 and Remark 3.5, $\{a, a + 8 + 16k_1, a + 4 + 8k_2\}$ is never a Diophantine triple with the property $D(n)$. □

Corollary 3.7. *If there exist positive integer m, k satisfy $0 < m < \frac{k-1}{2}$ and $\frac{3k(k-1)}{2m}$ is even, and $a = \frac{3k(k-1)}{2m} - (2k + 1) - 2m$, then $\{a, a + 4, a + 4k\}$ is a Diophantine triple with the property $D\left((4k - 1)^2 + 2a(2k - 1)\right)$.*

Proof. Suppose there exist positive integers m, k satisfying $0 < m < \frac{k-1}{2}$ and $\frac{3k(k-1)}{2m}$ is even. Let $P_b = k - 2m$, $P_c = 1$, $a = \frac{3k(k-1)}{2m} - 2k - 1 - 2m = \frac{3k^2 - 3k - 4mk - 2m - 4m^2}{2m}$, therefore P_b is a positive integer and a is an odd integer.
then $\frac{b(c-a)}{4P_b} + P_b$

$= \frac{(a+4)\cdot 4k}{4P_b} + P_b$
$= \frac{\left(3k^2-3k-4mk-2m-4m^2+8m\right)k}{2m(k-2m)} + k - 2m$
$= \frac{\left(3k^2-3k-4mk-4m^2+6m\right)k}{2m(k-2m)} + k - 2m$
$= \frac{(k-2m)(3k+2m-3)k}{2m(k-2m)} + k - 2m$
$= \frac{(3k+2m-3)k}{2m} + k - 2m$
$= \frac{3k^2+2mk-3k+2mk-4m^2}{2m}$
$= \frac{3k^2+4mk-3k-4m^2}{2m}$

On the other hand, $\frac{c(b-a)}{4P_c} + P_c$
$= \frac{(a+4k)\cdot 4}{4P_c} + P_c$
$= a + 4k + 1$
$= \frac{3k^2-3k-4mk-2m-4m^2}{2m} + 4k + 1$
$= \frac{3k^2-3k-4mk-2m-4m^2+8mk+2m}{2m}$
$= \frac{3k^2-3k+4mk-4m^2}{2m}$

Thus, $\frac{b(c-a)}{4P_b} + P_b = \frac{c(b-a)}{4P_c} + P_c$, and by Theorem 2.7,

$$n = \left(\frac{c\,(b-a)}{4P_c} + P_c\right)^2 - bc = (a + 4k + 1) - (a + 4)\,(a + 4k) = (4k - 1)^2 + 2a\,(2k - 1)$$

such that $\{a, a + 4, a + 4k\}$ is a Diophantine triple with the property $D\,(n)$. □

Example 3.8. *Suppose that $k = 5$ in Corollary 3.7. If there exists an integer n such that $\{a, a + 4, a + 20\}$ is a Diophantine triple with the property $D\,(n)$, then the integer m should be such that $\frac{3\cdot 5\cdot 4}{2\cdot m}$ is even and $0 < m < 2$. Thus $m = 1$, $a = 17$, $\{17, 21, 37\}$ is a Diophantine triple with the property $D\,(667)$.*

Remark 3.9. *For any positive integer m, $\{4m + 1, 4m + 5, 12m^2 + 20m + 5\}$ is a Diophantine triple with the property $D\,(n)$ where $n = 144m^4 + 432m^3 + 404m^2 + 120m + 11$.*

Proof. Let $P_b = m + 1$, $P_c = 1$,
then $\frac{b(c-a)}{4P_b} + P_b$
$= \frac{(4m+5)\left(12m^2+16m+4\right)}{4(m+1)} + (m + 1)$
$= (4m + 5)\,(3m + 1) + (m + 1)$
$= 12m^2 + 19m + 5 + m + 1$
$= 12m^2 + 20m + 6$
$\frac{c(b-a)}{4P_c} + P_c = 12m^2 + 20m + 6$
and $n = \left(\frac{c(b-a)}{4P_c} + P_c\right)^2 - bc$
$= 4(6m^2 + 10m + 3)^2 - (12m^2 + 20m + 5)\,(4m + 5)$
$= 144m^4 + 432m^3 + 404m^2 + 120m + 11.$

Therefore, $\{4m + 1, 4m + 5, 12m^2 + 20m + 5\}$ is a Diophantine triple with the property $D\,(n)$ where $n = 144m^4 + 432m^3 + 404m^2 + 120m + 11$. This is an infinite family of Diophantine triples $\{a, b, c\}$ when a, b and c are all odd. □

Example 3.10. *Let* $m = 3$*, then* $\{13, 17, 173\}$ *is a Diophantine triple with the property* $D\,(27335)$*. Let* $m = 4$*, then* $\{17, 21, 277\}$ *is a Diophantine triple with the property* $D\,(71467)$*.*

Remark 3.11. *For any positive integer* m*,* $\{4m + 3, 4m + 7, 4m^2 + 12m + 7\}$ *is a Diophantine triple with the property with the property* $D\,(n)$ *where* $n = 16m^4 + 80m^3 + 132m^2 + 80m + 15$*.*

Proof. Let $P_b = m + 1$, $P_c = 1$,
then $\frac{b(c-a)}{4P_b} + P_b$
$= \frac{(4m+7)\left(4m^2+8m+4\right)}{4(m+1)} + (m + 1)$
$= (4m + 7)\,(m + 1) + (m + 1)$
$= 4m^2 + 12m + 8$
$\frac{c(b-a)}{4P_c} + P_c = 4m^2 + 12m + 8$
and $n = \left(\frac{c(b-a)}{4P_c} + P_c\right)^2 - bc$
$= (4m^2 + 12m + 8)^2 - (4m + 7)\,(4m^2 + 12m + 7)$
$= 16m^4 + 80m^3 + 132m^2 + 80m + 15$

Therefore, $\{4m + 3, 4m + 7, 4m^2 + 12m + 7\}$ is a Diophantine triple with the property with the property $D\,(n)$ where $n = 16m^4 + 80m^3 + 132m^2 + 80m + 15$. □

Example 3.12. *Let* $m = 1$*, then* $\{7, 11, 23\}$ *is a Diophantine triple with the property* $D\,(323)$*. Let* $m = 4$*, then* $\{19, 23, 119\}$ *is a Diophantine triple with the property* $D\,(11663)$*.*

Corollary 3.13. *If* a *is an positive integer,* p_1 *and* p_2 *are two prime numbers with* $a+4 < p_1 < p_2$*, then for any integer* n*,* $\{a, a + 4, p_1, p_2\}$ *is never a Diophantine quadruple with the property* $D\,(n)$*.*

Proof. Suppose that p_1 and p_2 are two prime numbers with $a + 4 < p_1 < p_2$, if $\{a, a + 4, p_1, p_2\}$ is a Diophantine quadruple with the property $D\,(n)$, then $\{a, a + 4, p_1\}$ and $\{a, a + 4, p_1\}$ are both Diophantine triples with the property $D\,(n)$.

Since $\{a, a + 4, p_1\}$ is a Diophantine triple with the property $D\,(n)$,

$$\lambda = \frac{b\,(c-a)}{4P_b} + P_b = \frac{c\,(b-a)}{4P_c} + P_c \Rightarrow \lambda = \frac{(a+4)\,(p_1-a)}{4P_b} + P_b = \frac{p_1}{P_c} + P_c.$$

p_1 is a prime number implies that $P_c = 1$ or $P_c = p_1$, then $\lambda = p_1 + 1$, $n = \lambda^2 - bc = (p_1 + 1)^2 - p_1\,(a + 4)$.

Similarly, from $\{a, a + 4, p_2\}$ is a Diophantine triple with the property $D\,(n)$, we can get $n = (p_2 + 1)^2 - p_2\,(a + 4)$.

Thus, $(p_1 + 1)^2 - p_1\,(a + 4) = (p_2 + 1)^2 - p_2\,(a + 4)$, then we have
$(p_1 - p_2)\,(p_1 + p_2 - a - 2) = 0$, since $p_1 < p_2$, we obtain that $p_1 + p_2 = a + 2$, which is contradict to $a + 4 < p_1 < p_2$.

Therefore, for any integer n, $\{a, a + 4, p_1, p_2\}$ is never a Diophantine quadruple with the property $D\,(n)$. □

Theorem 3.14. *If positive odd integers a, A and B satisfy $a > AB$, $A|3a$, $B|(a+4)$ and $a \equiv AB \pmod 4$, then there exists an integer n such that the triple*

$$\left\{a, a+4, a+\frac{(a+4-AB)(3a+AB)}{4AB}\right\}$$

is a Diophantine triple with the property $D(n)$.

Proof. First of all, we can obtain that $4|a+4-AB$, $B|a+4-AB$ and $A|3a+AB$, then $4B|a+4-AB$, then $4AB|[(a+4-AB)(3a+AB)]$, thus $a+\frac{(a+4-AB)(3a+AB)}{4AB}$ is an integer.

According to Theorem 2.7, let $P_b = \frac{a+4-AB}{4}$, $P_c = 1$, then P_b is an integer.

Therefore $\frac{b(c-a)}{4P_b} + P_b$
$= \frac{(a+4)\cdot\frac{(a+4-AB)(3a+AB)}{4AB}}{4\cdot\frac{a+4-AB}{4}} + \frac{a+4-AB}{4}$
$= \frac{(a+4)(3a+AB)}{4AB} + \frac{a+4-AB}{4}$
$= \frac{3a^2+12a+aAB+4AB+aAB+4AB-(AB)^2}{4AB}$
$= \frac{3a^2+12a+2aAB+8AB-(AB)^2}{4AB}$
$= \frac{(a+4)(3a+2AB)-(AB)^2}{4AB} = \lambda$
and $\frac{c(b-a)}{4P_c} + P_c$
$= \frac{\left[a+\frac{(a+4-AB)(3a+AB)}{4AB}\right]\cdot 4}{4\cdot 1} + 1$
$= a + \frac{(a+4-AB)(3a+AB)}{4AB} + 1$
$= \frac{4aAB+3a^2+12a-3aAB+aAB+4AB-(AB)^2+4AB}{4AB}$
$= \frac{(a+4)(3a+2AB)-(AB)^2}{4AB}$
then $\frac{b(c-a)}{4P_b} + P_b = \frac{c(b-a)}{4P_c} + P_c$.

Then $n = \lambda^2 - bc$ with $b = a+4$ and $c = a + \frac{(a+4-AB)(3a+AB)}{4AB}$, and $\left\{a, a+4, a+\frac{(a+4-AB)(3a+AB)}{4AB}\right\}$ is a Diophantine triple with the property $D(n)$. We also note that a, $a+4$ and $a+\frac{(a+4-AB)(3a+AB)}{4AB}$ are all odd. □

Example 3.15. *If $a = 35$, then any element (A, B) in the set $\{(3,1), (7,1), (15,1), (1,3), (5,3)\}$ satisfies that $a > AB$, $A|3a$, $B|a+4$ and $a \equiv AB \pmod 4$. Thus triples $\{35, 39, 359\}$, $\{35, 39, 163\}$ and $\{35, 39, 83\}$ are Diophantine triples with the property $D(115599)$, $D(20539)$ and $D(3819)$ respectively.*

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