Notes on Number Theory and Discrete Mathematics
ISSN 1310–5132
Vol. 21, 2015, No. 3, 80–87

On the density of ranges of generalized divisor functions

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Abstract: The range of the divisor function σ_{-1} is dense in the interval $[1, \infty)$. However, although the range of the function σ_{-2} is a subset of the interval $\left[1, \frac{\pi^2}{6}\right)$, we will see that the range of σ_{-2} is not dense in $\left[1, \frac{\pi^2}{6}\right)$. We begin by generalizing the divisor functions to a class of functions σ_t for all real t. We then define a constant $\eta \approx 1.8877909$ and show that if $r \in (1, \infty)$, then the range of the function σ_{-r} is dense in the interval $[1, \zeta(r))$ if and only if $r \leq \eta$. We end with an open problem.
Keywords: Density, Divisor function.
AMS Classification: Primary 11B05; Secondary 11A25.

1 Introduction

Throughout this paper, we will let $\mathbb{N}$ denote the set of positive integers, and we will let p_i denote the i^{th} prime number.

For any integer t, the divisor function σ_t is a multiplicative arithmetic function defined by $\sigma_t(n) = \sum_{\substack{d|n \\ d>0}} d^t$ for all positive integers n. The value of $\sigma_1(n)$ is the sum of the positive divisors of n, while the value of $\sigma_0(n)$ is simply the number of positive divisors of n.

Another interesting divisor function is σ_{-1}, which is often known as the abundancy index. One may show [2] that the range of σ_{-1} is a subset of the interval $[1,\infty)$ that is dense in $[1,\infty)$. If $t<-1$, then the range of σ_t is a subset of the interval $[1,\zeta(-t))$, where ζ denotes the Riemann zeta function. This is because, for any positive integer n, $\sigma_t(n)=\sum_{\substack{d|n\\d>0}} d^t < \sum_{i=1}^{\infty} i^t = \zeta(-t)$. For example, the range of the function σ_{-2} is a subset of the interval $\left[1,\frac{\pi^2}{6}\right)$. However, it is interesting to note that the range of the function σ_{-2} is not dense in the interval $\left[1,\frac{\pi^2}{6}\right)$. To see this, let n be a positive integer. If $2|n$, then $\sigma_{-2}(n)\geq \frac{1}{1^2}+\frac{1}{2^2}=\frac{5}{4}$. On the other hand, if $2\nmid n$, then $\sigma_{-2}(n) < \sum_{d\in\mathbb{N}\setminus(2\mathbb{N})}\frac{1}{d^2} = \frac{\zeta(2)}{\left(\frac{1}{1-2^{-2}}\right)} = \frac{\pi^2}{8}$. As $\frac{\pi^2}{8}<\frac{5}{4}$, we see that there is a "gap" in the range of σ_{-2}. In other words, there are no positive integers n such that $\sigma_{-2}(n)\in\left(\frac{\pi^2}{8},\frac{5}{4}\right)$.

Our first goal is to generalize the divisor functions to allow for nonintegral subscripts. For example, we might consider the function $\sigma_{-\sqrt{2}}$, defined by $\sigma_{-\sqrt{2}}(n)=\sum_{\substack{d|n\\d>0}} d^{-\sqrt{2}}$. We formalize this idea in the following definition.

Definition 1.1. For a real number t, define the function $\sigma_t\colon\mathbb{N}\to\mathbb{R}$ by $\sigma_t(n)=\sum_{\substack{d|n\\d>0}} d^t$ for all $n\in\mathbb{N}$. Also, we will let $\log\sigma_t=\log\circ\sigma_t$.

In analyzing the ranges of these generalized divisor functions, we will find a constant which serves as a "boundary" between divisor functions with dense ranges and divisor functions with ranges that have gaps. Note that, for any real number t, we may write $\sigma_t=I_0*I_t$, where I_0 and I_t are arithmetic functions defined by $I_0(n)=1$ and $I_t(n)=n^t$. As I_0 and I_t are multiplicative, we find that σ_t is multiplicative.

2 The ranges of functions σ_{-r}

Theorem 2.1. *Let r be a real number greater than 1. The range of σ_{-r} is dense in the interval $[1,\zeta(r))$ if and only if $1+\frac{1}{p_m^r}\leq\prod_{i=m+1}^{\infty}\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right)$ for all positive integers m.*

Proof. First, suppose that $1+\frac{1}{p_m^r}\leq\prod_{i=m+1}^{\infty}\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right)$ for all positive integers m. We will show that the range of $\log\sigma_{-r}$ is dense in the interval $[0,\log(\zeta(r)))$, which will imply that the range of σ_{-r} is dense in $[1,\zeta(r))$. Choose some arbitrary $x\in(0,\log(\zeta(r)))$, and define $X_0=0$. For each positive integer n, we define α_n and X_n in the following manner. If $X_{n-1} + \log\left(\sum_{j=0}^{\infty} \frac{1}{p_n^{jr}}\right) \le x$, define $\alpha_n = -1$. If $X_{n-1} + \log\left(\sum_{j=0}^{\infty} \frac{1}{p_n^{jr}}\right) > x$, define α_n to be the largest nonnegative integer that satisfies $X_{n-1} + \log\left(\sum_{j=0}^{\alpha_n} \frac{1}{p_n^{jr}}\right) \le x$. Define X_n by

$$X_n = \begin{cases} X_{n-1} + \log\left(\sum_{j=0}^{\alpha_n} \frac{1}{p_n^{jr}}\right), & \text{if } \alpha_n \ge 0; \\ X_{n-1} + \log\left(\sum_{j=0}^{\infty} \frac{1}{p_n^{jr}}\right), & \text{if } \alpha_n = -1. \end{cases}$$

Also, for each $n \in \mathbb{N}$, define D_n by

$$D_n = \begin{cases} \log\left(\sum_{j=0}^{\infty} \frac{1}{p_n^{jr}}\right) - \log\left(\sum_{j=0}^{\alpha_n} \frac{1}{p_n^{jr}}\right), & \text{if } \alpha_n \ge 0; \\ 0, & \text{if } \alpha_n = -1, \end{cases}$$

and let $E_n = \sum_{i=1}^{n} D_i$. Note that

$$\lim_{n\to\infty}(X_n + E_n) = \lim_{n\to\infty}\left(X_n + \sum_{i=1}^{n} D_i\right)$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\log\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right) = \log(\zeta(r)).$$

Now, because the sequence $(X_n)_{n=1}^{\infty}$ is bounded and monotonic, we know that there exists some real number γ such that $\lim_{n\to\infty} X_n = \gamma$. We wish to show that $\gamma = x$.

Notice that we defined the sequence $(X_n)_{n=1}^{\infty}$ so that $X_n \le x$ for all $n \in \mathbb{N}$. Hence, we know that $\gamma \le x$. Now, suppose $\gamma < x$. Then $\lim_{n\to\infty} E_n = \log(\zeta(r)) - \gamma > \log(\zeta(r)) - x$. This implies that there exists some positive integer N such that $E_n > \log(\zeta(r)) - x$ for all integers $n \ge N$. Let m be the smallest positive integer that satisfies $E_m > \log(\zeta(r)) - x$. If $\alpha_m = -1$ and $m > 1$, then $D_m = 0$, so $E_{m-1} = E_m > \log(\zeta(r)) - x$. However, this contradicts the minimality of m. If $\alpha_m = -1$ and $m = 1$, then $0 = D_m = E_m > \log(\zeta(r)) - x$, which is also a contradiction. Thus, we conclude that $\alpha_m \ge 0$. This means that $X_m + D_m = X_{m-1} + \log\left(\sum_{j=0}^{\infty}\frac{1}{p_m^{jr}}\right) > x$, so $D_m > x - X_m$. Furthermore,

$$\log\left(\prod_{i=m+1}^{\infty}\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right)\right) = \sum_{i=m+1}^{\infty}\log\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right)$$
$$= \log(\zeta(r)) - \sum_{i=1}^{m}\log\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right)$$

$$= \log(\zeta(r)) - E_m - X_m < x - X_m < D_m, \tag{1}$$

and we originally assumed that $1 + \frac{1}{p_m^r} \le \prod_{i=m+1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right)$. This means that

$\log\left(1 + \frac{1}{p_m^r}\right) < D_m = \log\left(\sum_{j=0}^{\infty} \frac{1}{p_m^{jr}}\right) - \log\left(\sum_{j=0}^{\alpha_m} \frac{1}{p_m^{jr}}\right)$, or, equivalently,

$\log\left(1 + \frac{1}{p_m^r}\right) + \log\left(\sum_{j=0}^{\alpha_m} \frac{1}{p_m^{jr}}\right) < \log\left(\frac{p_m^r}{p_m^r - 1}\right)$. If $\alpha_m > 0$, we have

$$\log\left(\left(1 + \frac{1}{p_m^r}\right)^2\right) \le \log\left(1 + \frac{1}{p_m^r}\right) + \log\left(\sum_{j=0}^{\alpha_m} \frac{1}{p_m^{jr}}\right) < \log\left(\frac{p_m^r}{p_m^r - 1}\right),$$

so $\left(1 + \frac{1}{p_m^r}\right)^2 < \frac{p_m^r}{p_m^r - 1}$. We may write this as $1 + \frac{2}{p_m^r} + \frac{1}{p_m^{2r}} < 1 + \frac{1}{p_m^r - 1}$, so $2 < \frac{p_m^r}{p_m^r - 1} = 1 + \frac{1}{p_m^r - 1}$. As $p_m^r > 2$, this is a contradiction. Hence, $\alpha_m = 0$. By the definitions of α_m and X_m, this implies that $X_{m-1} + \log\left(1 + \frac{1}{p_m^r}\right) > x$ and that $X_m = X_{m-1}$. Therefore, $\log\left(1 + \frac{1}{p_m^r}\right) > x - X_{m-1} = x - X_m$. However, recalling from (1) that

$$\sum_{i=m+1}^{\infty} \log\left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}}\right) < x - X_m,$$

we find that

$$\sum_{i=m+1}^{\infty} \log\left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}}\right) < \log\left(1 + \frac{1}{p_m^r}\right),$$

which is a contradiction because we originally assumed that $1 + \frac{1}{p_m^r} \le \prod_{i=m+1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right)$. Therefore, $\gamma = x$.

We now know that $\lim_{n\to\infty} X_n = x$. To show that the range of $\log \sigma_{-r}$ is dense in $[0, \log(\zeta(r)))$, we need to construct a sequence $(C_n)_{n=1}^{\infty}$ of elements of the range of $\log \sigma_{-r}$ that satisfies $\lim_{n\to\infty} C_n = x$. We do so in the following fashion. For each positive integer n, write

$$Y_n = \begin{cases} 1, & \text{if } \alpha_n \ge 0; \\ 0, & \text{if } \alpha_n = -1, \end{cases}$$

$$Z_n = \begin{cases} 0, & \text{if } \alpha_n \ge 0; \\ 1, & \text{if } \alpha_n = -1, \end{cases}$$

and

$$\beta_n = \begin{cases} \alpha_n, & \text{if } \alpha_n \ge 0; \\ 0, & \text{if } \alpha_n = -1. \end{cases}$$

Now, for each positive integer n, define C_n by

$$C_n = \sum_{k=1}^{n} \left(Y_k \log \left(\sum_{j=0}^{\beta_k} \frac{1}{p_k^{jr}} \right) + Z_k \log \left(\sum_{j=0}^{n} \frac{1}{p_k^{jr}} \right) \right).$$

Notice that, by the way we defined X_n, we have

$$X_n = \sum_{k=1}^{n} \left(Y_k \log \left(\sum_{j=0}^{\beta_k} \frac{1}{p_k^{jr}} \right) + Z_k \log \left(\sum_{j=0}^{\infty} \frac{1}{p_k^{jr}} \right) \right).$$

Therefore, $\lim\limits_{n\to\infty} C_n = \lim\limits_{n\to\infty} X_n = x$. All we need to do now is show that each C_n is in the range of $\log \sigma_{-r}$. We have

$$\begin{aligned} C_n &= \sum_{k=1}^{n} \left(Y_k \log \left(\sum_{j=0}^{\beta_k} \frac{1}{p_k^{jr}} \right) + Z_k \log \left(\sum_{j=0}^{n} \frac{1}{p_k^{jr}} \right) \right) \\ &= \sum_{\substack{k\in\mathbb{N} \\ k\le n \\ \alpha_k \ge 0}} \log \left(\sum_{j=0}^{\alpha_k} \frac{1}{p_k^{jr}} \right) + \sum_{\substack{k\in\mathbb{N} \\ k\le n \\ \alpha_k = -1}} \log \left(\sum_{j=0}^{n} \frac{1}{p_k^{jr}} \right) \\ &= \log \left[\left(\prod_{\substack{k\in\mathbb{N} \\ k\le n \\ \alpha_k \ge 0}} \sigma_{-r}(p_k^{\alpha_k}) \right) \left(\prod_{\substack{k\in\mathbb{N} \\ k\le n \\ \alpha_k = -1}} \sigma_{-r}(p_k^{n}) \right) \right] \\ &= \log \sigma_{-r} \left(\left(\prod_{\substack{k\in\mathbb{N} \\ k\le n \\ \alpha_k \ge 0}} p_k^{\alpha_k} \right) \left(\prod_{\substack{k\in\mathbb{N} \\ k\le n \\ \alpha_k \ge 0}} p_k^{n} \right) \right). \end{aligned}$$

We finally conclude that if $1 + \frac{1}{p_m^r} \le \prod_{i=m+1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right)$ for all positive integers m, then the range of σ_{-r} is dense in the interval $[1, \zeta(r))$.

Conversely, suppose that there exists some positive integer m such that $1 + \frac{1}{p_m^r} > \prod_{i=m+1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right)$. Fix some $N \in \mathbb{N}$, and let $N = \prod_{i=1}^{v} q_i^{\gamma_i}$ be the canonical prime factorization of N. If $p_s | N$ for some $s \in \{1, 2, \ldots, m\}$, then

$$\sigma_{-r}(N) \ge 1 + \frac{1}{p_s^r} \ge 1 + \frac{1}{p_m^r}.$$

On the other hand, if $p_s \nmid N$ for all $s \in \{1, 2, \ldots, m\}$, then

$$\sigma_{-r}(N) = \prod_{i=1}^{v} \sigma_{-r}(q_i^{\gamma_i}) = \prod_{i=1}^{v} \left(\sum_{j=0}^{\gamma_i} \frac{1}{q_i^{jr}} \right)$$

$$< \prod_{i=1}^{v} \left(\sum_{j=0}^{\infty} \frac{1}{q_i^{jr}} \right) < \prod_{i=m+1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right).$$

Because N was arbitrary, this shows that there is no element of the range of σ_{-r} in the interval $\left[\prod_{i=m+1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right), 1 + \frac{1}{p_m^r} \right)$, which means that the range of σ_{-r} is not dense in $[1, \zeta(r))$. □

Theorem 2.1 provides us with a method to determine values of $r > 1$ with the property that the range of σ_{-r} is dense in $[1, \zeta(r))$. However, doing so is still a somewhat difficult task. Luckily, for $r \in (1, 2]$, we may greatly simplify the problem with the help of the following theorem. First, we need a short lemma.

Lemma 2.1. *If* $j \in \mathbb{N} \backslash \{1, 2, 4\}$*, then* $\frac{p_{j+1}}{p_j} < \sqrt{2}$.

Proof. Pierre Dusart [1] has shown that, for $x \geq 396\,738$, there must be at least one prime in the interval $\left[x, x + \frac{x}{25 \log^2 x} \right]$. Therefore, whenever $p_j > 396\,738$, we may set $x = p_j + 1$ to get $p_{j+1} \leq (p_j + 1) + \frac{p_j + 1}{25 \log^2(p_j + 1)} < \sqrt{2} p_j$. Using Mathematica 9.0 [3], we may quickly search through all the primes less than $396\,738$ to conclude the desired result. □

Theorem 2.2. *Let* r *be a real number in the interval* $(1, 2]$*. The range of* σ_{-r} *is dense in the interval* $[1, \zeta(r))$ *if and only if* $1 + \frac{1}{p_m^r} \leq \prod_{i=m+1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right)$ *for all* $m \in \{1, 2, 4\}$.

Proof. Let $F(m, r) = \left(1 + \frac{1}{p_m^r} \right) \prod_{i=1}^{m} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right)$ so that the inequality $1 + \frac{1}{p_m^r} \leq \prod_{i=m+1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right)$ is equivalent to $F(m, r) \leq \zeta(r)$. In light of Theorem 2.1, it suffices to show that if $F(m, r) \leq \zeta(r)$ for all $m \in \{1, 2, 4\}$, then $F(m, r) \leq \zeta(r)$ for all $m \in \mathbb{N}$. Thus, let us assume that r is such that $F(m, r) \leq \zeta(r)$ for all $m \in \{1, 2, 4\}$. If $m \in \mathbb{N} \backslash \{1, 2, 4\}$, then Lemma 2.1 tells us that $\frac{p_{m+1}}{p_m} < \sqrt{2} \leq \sqrt[r]{2}$, which implies that $\frac{2}{p_{m+1}^r} > \frac{1}{p_m^r}$. We then have

$$F(m+1, r) = \left(1 + \frac{1}{p_{m+1}^r} \right) \prod_{i=1}^{m+1} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right) > \left(1 + \frac{1}{p_{m+1}^r} \right)^2 \prod_{i=1}^{m} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right)$$

$$> \left(1 + \frac{2}{p_{m+1}^r} \right) \prod_{i=1}^{m} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right) > \left(1 + \frac{1}{p_m^r} \right) \prod_{i=1}^{m} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right) = F(m, r)$$

for all $m \in \mathbb{N} \backslash \{1, 2, 4\}$. This means that $F(3, r) < F(4, r) \leq \zeta(r)$. Furthermore, $F(m, r) < \zeta(r)$ for all integers $m \geq 5$ because $(F(m, r))_{m=5}^{\infty}$ is a strictly increasing sequence and $\lim_{m \to \infty} F(m, r) = \zeta(r)$. □

We have seen that, for $r \in (1,2]$, the range of σ_{-r} is dense in $[1,\zeta(r))$ if and only if $F(m,r) \leq \zeta(r)$ for all $m \in \{1,2,4\}$. Using Mathematica 9.0, one may plot a function $g_m(r) = F(m,r) - \zeta(r)$ for each $m \in \{1,2,4\}$. It is then easy to verify that g_2 has precisely one root, say η, in the interval $(1,2]$ (for anyone seeking a more rigorous proof of this fact, we mention that it is fairly simple to show that $g_2'(r) > 0$ for all $r \in (1,2]$). Furthermore, one may confirm that $g_1(r), g_2(r), g_4(r) \leq 0$ for all $r \in (1,\eta]$ and that $g_2(r) > 0$ for all $r \in (\eta, 3]$. Hence, we have proven (or at least left the reader to verify) the first part of the following theorem.

Theorem 2.3. *Let η be the unique number in the interval $(1,2]$ that satisfies the equation*

$$\left(\frac{2^\eta}{2^\eta - 1}\right)\left(\frac{3^\eta + 1}{3^\eta - 1}\right) = \zeta(\eta).$$

If $r \in (1,\infty)$, then the range of the function σ_{-r} is dense in the interval $[1,\zeta(r))$ if and only if $r \leq \eta$.

Proof. In virtue of the preceding paragraph, we know from the fact that

$$g_2(\eta) = F(2,\eta) - \zeta(\eta) = \left(\frac{2^\eta}{2^\eta - 1}\right)\left(\frac{3^\eta + 1}{3^\eta - 1}\right) - \zeta(\eta) = 0$$

that if $r \in (1,3]$, then the range of σ_{-r} is dense in $[1,\zeta(r))$ if and only if $r \leq \eta$. We now show that the range of σ_{-r} is not dense in $[1,\zeta(r))$ if $r > 3$. To do so, we merely need to show that $F(1,r) > \zeta(r)$ for all $r > 3$. For $r > 3$, we have

$$F(1,r) = \left(1 + \frac{1}{2^r}\right)\sum_{j=0}^{\infty}\frac{1}{2^{jr}} > \left(1 + \frac{1}{2^r}\right)^2 = 1 + \frac{1}{2^r} + \frac{3}{4}\left(\frac{1}{2^{r-1}}\right)$$

$$> 1 + \frac{1}{2^r} + \frac{1}{(r-1)2^{r-1}} = 1 + \frac{1}{2^r} + \int_2^\infty \frac{1}{x^r}dx > \zeta(r).$$

□

3 An open problem

We end by acknowledging that it might be of interest to consider the number of "gaps" in the range of σ_{-r} for various r. For example, for which values of $r \in (1,\infty)$ is there precisely one gap in the range of σ_{-r}? More generally, if we are given a positive integer L, then, for what values of $r > 1$ is the closure of the range of σ_{-r} a union of exactly L disjoint subintervals of $[1,\zeta(r)]$?

Acknowledgements

Dedicated to Mrs. Robin Wiseman, an amazing teacher who always told me I should write a book and dedicate it to her. This is not a book, but I think it should still count.

The author would also like to thank Professor Pete Johnson for inviting him to the 2014 REU Program in Algebra and Discrete Mathematics at Auburn University.

This work was supported by National Science Foundation grant no. 1262930.

References

[1] Dusart, P. (2010) Estimates of some functions over primes without R.H., arXiv:1002.0442.

[2] Laatsch, R. (1986) Measuring the abundancy of integers. *Math. Mag.*, 59(2), 84–92.

[3] Wolfram Research, Inc., Mathematica, Version 9.0, Champaign, IL (2012).