Notes on Number Theory and Discrete Mathematics
ISSN 1310–5132
21, 2015, No. 3, 56–63

Tridiagonal matrices related to subsequences of balancing and Lucas-balancing numbers

Prasanta K Ray[1] and Gopal K Panda[2]

[1] Department of Mathematics, Veer Surendra Sai University of Technology
Odisha, Burla-768018, India
e-mail: rayprasanta2008@gmail.com

[2] Department of Mathematics, National Institute of Technology Rourkela
Rourkela-769008, India
e-mail: gkpanda@nitrkl.ac.in

Abstract: It is well known that balancing and Lucas-balancing numbers are expressed as determinants of suitable tridiagonal matrices. The aim of this paper is to express certain subsequences of balancing and Lucas-balancing numbers in terms of determinants of tridiagonal matrices. Using these tridiagonal matrices, a factorization of the balancing numbers is also derived.
Keywords: Balancing numbers, Balancers, Lucas-balancing numbers, Tridiagonal matrices.
AMS Classification: 11B39, 11B83.

1 Introduction

The concept of balancing numbers was originally introduced by Behera et.al [1] in connection with the Diophantine equation

$$1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

where, they call n a balancing number and r the balancer corresponds to n. The sequence of balancing number $\{B_n\}$ satisfy the recurrence relations

$$B_{n+1} = 6B_n - B_{n-1},\ n \geq 2,$$

with $B_1 = 1, B_2 = 6$ and $B_{n+1} = \frac{{B_n}^2-1}{B_{n-1}},\ n \geq 2$. The number sequence closely associates to the balancing numbers is the sequence of Lucas-balancing numbers whose recurrence relation is

$$C_{n+1} = 6C_n - C_{n-1},\ n \geq 2,$$

with $C_1 = 3, C_2 = 17$, where C_n denotes the n^{th} Lucas-balancing number. The closed form (popularly known as Binet's formula) for both balancing and Lucas-balancing numbers are respectively given by $B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$ and $C_n = \frac{\lambda_1^n + \lambda_2^n}{2}$ where $\lambda_1 = 3 + \sqrt{8}$, $\lambda_2 = 3 - \sqrt{8}$ [1, 7].

Panda [8] has shown that, the Lucas-balancing numbers are associated with balancing numbers in the way Lucas numbers are associated with Fibonacci numbers. In [7], Panda and Ray have proved that the Lucas-balancing numbers are nothing but the even ordered terms of the associated Pell sequence. Also they have shown that the n^{th} balancing number is product of n^{th} Pell number and n^{th} associated Pell numbers. Liptai, et al. [5] added another interesting result to the theory of balancing numbers by generalizing these numbers. A. Berczes, et al. [3] and P. Olajos [6] surveyed some interesting properties and results on generalized balancing numbers. Recently, many interesting results on balancing numbers and their related sequences are studied by different authors [2, 4, 10, 12, 13, 14, 15, 16, 17]. There is another way to generate balancing numbers through matrices called as balancing matrices which were introduced in [11].

There are many connections between balancing, Lucas-balancing numbers and tridiagonal matrices. In [9], Ray introduced a family of tridiagonal matrices of order n

$$D(n) = \begin{pmatrix} 6 & -i & & & \\ i & 6 & -i & & \\ & i & 6 & \ddots & \\ & & \ddots & \ddots & -i \\ & & & i & 6 \end{pmatrix}, \tag{1}$$

whose determinants $|D(k-1)|$ are nothing but the balancing numbers B_k, starting with $k = 2$. Replacing the first row first column entry of $D(n)$ by 3 to get

$$M(n) = \begin{pmatrix} 3 & -i & & & \\ i & 6 & -i & & \\ & i & 6 & \ddots & \\ & & \ddots & \ddots & -i \\ & & & i & 6 \end{pmatrix}, \tag{2}$$

whose determinants now generate the Lucas-balancing numbers C_k, starting with $k = 1$. In this paper, we extend these results to construct families of tridiagonal matrices whose determinants generate any arbitrary linear sub-sequences $B_{\alpha k+\beta}$ or $C_{\alpha k+\beta}$ of the balancing and Lucas-balancing numbers with $k \in \mathbb{Z}^+$. We choose a specific linear subsequence of balancing numbers and use it to derive the factorization

$$B_{2mn} = B_{2m} \prod_{1 \le k \le n-1} 2\left(C_{2m} - \cos\frac{k\pi}{n}\right). \tag{3}$$

The factorization (3) indeed, a generalization of one of

$$B_n = \prod_{1 \le k \le n-1} 2\left(3 - \cos\frac{k\pi}{n}\right)$$

given in [9]. In order to derive (3), we present the following known result which describes the sequence of determinants for a general tridiagonal matrix (see Theorem 1.1, [9]).

Theorem 1.1. *If the family of tridiagonal matrices* $A(n),\ n = 1, 2, \ldots$ *is of the form*

$$A(n) = \begin{pmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & a_{32} & a_{33} & \ddots & \\ & & \ddots & \ddots & a_{(n-1)n} \\ & & & a_{n(n-1)} & a_{nn} \end{pmatrix},$$

then, the successive determinants of A_n *are given by the recursive formulas:*

$$\begin{aligned} |A(1)| &= a_{11} \\ |A(2)| &= a_{11}a_{22} - a_{12}a_{21} \\ |A(n)| &= a_{nn}|A(n-1)| - a_{(n-1)n}a_{n(n-1)}|A(n-2)|. \end{aligned}$$

2 Balancing subsequences

By virtue of Theorem 1.1, we can generalize the families of tridiagonal matrices given by (1) and (2). For every linear subsequence of balancing numbers, we construct a family of tridiagonal matrices whose successive determinants are shown in the following result.

Theorem 2.1. *The symmetric tridiagonal family of matrices* $D_{\alpha,\beta}(k)$, $k = 1, 2, \ldots$, *whose entries are*

$$\begin{aligned} &d_{1,1} = B_{\alpha+\beta},\ d_{2,2} = \left\lceil \frac{B_{2\alpha+\beta}}{B_{\alpha+\beta}} \right\rceil; d_{j,j} = 2C_\alpha,\ \ 3 \le j \le k, \\ &d_{1,2} = d_{2,1} = \sqrt{d_{2,2}B_{\alpha+\beta} - B_{2\alpha+\beta}}; d_{j,j+1} = d_{j+1,j} = 1,\ \ 2 \le j \le k \end{aligned}$$

with positive integers α *and natural numbers* β*, has successive determinants* $|D_{\alpha,\beta}(k)| = B_{\alpha k+\beta}$.

In order to prove Theorem 2.1, we need the following lemma.

Lemma 2.2. *For all positive integers* k *and* n, $B_{k+n} = 2C_nB_k - B_{k-n}$.

Proof. Using Binet's formulas for B_n, C_n and since $\lambda_1\lambda_2 = 1$, we obtain

$$\begin{aligned} 2C_nB_k - B_{k-n} &= (\lambda_1^n + \lambda_2^n)\frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2} - \frac{\lambda_1^{k-n} - \lambda_2^{k-n}}{\lambda_1 - \lambda_2} \\ &= \frac{\lambda_1^{n+k} - \lambda_1^n\lambda_2^k + \lambda_1^k\lambda_2^n - \lambda_2^{n+k} - \lambda_1^{k-n} + \lambda_2^{k-n}}{\lambda_1 - \lambda_2} \\ &= \frac{\lambda_1^{n+k} - \lambda_2^{n+k}}{\lambda_1 - \lambda_2} = B_{k+n}, \end{aligned}$$

which completes the proof. □

Now we are in a position to prove Theorem 2.1.

Proof. (Theorem 2.1) We use method of induction to prove this result. Clearly, result holds for $k = 1, 2,$

$$|D_{\alpha,\beta}(1)| = B_{\alpha+\beta}$$
$$|D_{\alpha,\beta}(2)| = \begin{vmatrix} B_{\alpha+\beta} & \sqrt{d_{2,2}B_{\alpha+\beta} - B_{2\alpha+\beta}} \\ \sqrt{d_{2,2}B_{\alpha+\beta} - B_{2\alpha+\beta}} & \left\lceil \frac{B_{2\alpha+\beta}}{B_{\alpha+\beta}} \right\rceil \end{vmatrix} = B_{2\alpha+\beta}.$$

Assume that, $|D_{\alpha,\beta}(k)| = B_{\alpha k+\beta}$ for any natural number less than or equal to k. Then by virtue of Theorem 1.1 and Lemma 2.2, we get

$$\begin{aligned} |D_{\alpha,\beta}(k+1)| &= d_{k,k}|D_{\alpha,\beta}(k)| - d_{k,k-1}d_{k-1,k}|D_{\alpha,\beta}(k-1)| \\ &= 2C_\alpha|D_{\alpha,\beta}(k)| - |D_{\alpha,\beta}(k-1)| \\ &= 2C_\alpha B_{\alpha k+\beta} - B_{\alpha(k-1)+\beta} = B_{\alpha(k+1)+\beta}, \end{aligned}$$

which ends the proof. □

By Theorem 2.1, one can construct a family of tridiagonal matrices whose successive determinants form any linear subsequence of balancing numbers. For example, the determinants of the following tridiagonal matrices

$$\begin{pmatrix} 6 & 0 & & & \\ 0 & 1155 & 1 & & \\ & 1 & 1154 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 1154 \end{pmatrix}, \begin{pmatrix} 6930 & \sqrt{35} & & & \\ \sqrt{35} & 198 & 1 & & \\ & 1 & 198 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 198 \end{pmatrix},$$
$$\begin{pmatrix} 40391 & \sqrt{1189} & & & \\ \sqrt{1189} & 34 & 1 & & \\ & 1 & 34 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 34 \end{pmatrix}$$

are respectively given by B_{4k-2}, B_{3k+3} and B_{2k+5}.

3 Lucas-balancing subsequences

In the last section, we discussed the link between linear balancing subsequences and tridiagonal matrices. In a similar manner, we can generalize the family of tridiagonal matrix (2) for linear subsequences of Lucas-balancing numbers. The following theorem demonstrates the claim.

Theorem 3.1. *For every natural number k, the symmetric tridiagonal family of matrices $M_{\alpha,\beta}(k)$, whose entries are*

$$m_{1,1} = C_{\alpha+\beta},\ m_{2,2} = \left\lceil \frac{C_{2\alpha+\beta}}{C_{\alpha+\beta}} \right\rceil ; m_{j,j} = 2C_{\alpha},\ \ 3 \leq j \leq k,$$
$$m_{1,2} = m_{2,1} = \sqrt{m_{2,2}C_{\alpha+\beta} - C_{2\alpha+\beta}}\, ; m_{j,j+1} = m_{j+1,j} = 1,\ \ 2 \leq j \leq k$$

with positive integers α and natural numbers β, has successive determinants $|M_{\alpha,\beta}(k)| = C_{\alpha k+\beta}$.

Following result is required to prove the theorem. The proof of the lemma is omitted as it is analogous to Lemma 2.2.

Lemma 3.2. *For any integers k and n,*

$$C_{k+n} = 2C_nC_k - C_{k-n}.$$

Proof. (Theorem 3.1) Once again, method of induction comes into the picture. Clearly, the result is true for $k = 1, 2$, since

$$|M_{\alpha,\beta}(1)| = C_{\alpha+\beta}$$
$$|M_{\alpha,\beta}(2)| = \begin{vmatrix} C_{\alpha+\beta} & \sqrt{m_{2,2}C_{\alpha+\beta} - C_{2\alpha+\beta}} \\ \sqrt{m_{2,2}C_{\alpha+\beta} - C_{2\alpha+\beta}} & \left\lceil \frac{C_{2\alpha+\beta}}{C_{\alpha+\beta}} \right\rceil \end{vmatrix} = C_{2\alpha+\beta}.$$

Assume that, $|M_{\alpha,\beta}(k)| = C_{\alpha k+\beta}$ for $1 \leq k \leq \mathbb{N}$. Then, by Theorem 1.1 and Lemma 3.2, we have

$$\begin{aligned} |M_{\alpha,\beta}(k+1)| &= m_{k,k}|M_{\alpha,\beta}(k)| - m_{k,k-1}m_{k-1,k}|M_{\alpha,\beta}(k-1)| \\ &= 2C_{\alpha}|M_{\alpha,\beta}(k)| - |M_{\alpha,\beta}(k-1)| \\ &= 2C_{\alpha}C_{\alpha k+\beta} - C_{\alpha(k-1)+\beta} = C_{\alpha(k+1)+\beta}. \end{aligned}$$

which end the proof. □

Using Theorem 3.1, one can construct a family of tridiagonal matrices whose successive determinants form any linear subsequence of Lucas-balancing numbers. For example, the determinants of the following tridiagonal matrices

$$\begin{pmatrix} 17 & 0 & & & \\ 0 & 1153 & 1 & & \\ & 1 & 1154 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 1154 \end{pmatrix}, \begin{pmatrix} 19601 & \sqrt{99} & & & \\ \sqrt{99} & 198 & 1 & & \\ & 1 & 198 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 198 \end{pmatrix},$$
$$\begin{pmatrix} 114243 & \sqrt{3363} & & & \\ \sqrt{3363} & 34 & 1 & & \\ & 1 & 34 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 34 \end{pmatrix}$$

are respectively given by C_{4k-2}, C_{3k+3} and C_{2k+5}.

4 Factorization of balancing numbers using tridiagonal matrices

In this section, we establish the factorization (3) presented in section 1. We consider the following symmetric tridiagonal matrices $T_m(n),\ n = 1, 2, \dots$ where

$$T_m(n) = \begin{pmatrix} 2C_{2m}B_{2m} & \sqrt{B_{2m}} & & & \\ \sqrt{B_{2m}} & 2C_{2m} & 1 & & \\ & 1 & 2C_{2m} & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2C_{2m} \end{pmatrix}.$$

In view of Lemma 2.2 and Lemma 3.2, we have, $2C_{2m}B_{2m} = B_{4m}$ and

$$\left\lceil \frac{B_{6m}}{B_{4m}} \right\rceil = \left\lceil \frac{2B_{4m}C_{2m} - B_{2m}}{B_{4m}} \right\rceil = \lceil 2C_{2m} - B_{2m} \rceil = 2C_{2m}.$$

Also,

$$\sqrt{\left\lceil \frac{B_{6m}}{B_{4m}} \right\rceil B_{4m} - B_{6m}} = \sqrt{2C_{2m}B_{4m} - B_{6m}} = \sqrt{B_{2m}}.$$

Thus, $T_m(n)$ is a specific example of the family of tridiagonal matrices $D_{\alpha,\beta}(n)$ with $\alpha = 2m$ and $\beta = 2m$. By Theorem 2.1, we get $|T_m(n)| = B_{2m(n+1)}$. If e_j be the j^{th} column of an $n \times n$ identity matrix, we have

$$|T_m(n)| = B_{2m}|R_m(n)|,$$

where $R_m(n) = \left(I + (\frac{1}{B_{2m}} - 1)e_1 e_1^T\right) T_m(n)$. If $\lambda_k, k = 1, 2, \dots$ be the eigenvalues of $R_m(n)$ with corresponding eigenvectors X_k and since the product of all eigenvalues is nothing but the value of the determinant, we get $|R_m(n)| = \prod_{k=1}^{n} \lambda_k$. We now introduce a new tridiagonal matrix $S_m(n) = R_m(n) - 2C_{2m}I$ and observe that,

$$S_m(n)X_k = R_m(n)X_k - 2C_{2m}IX_k = \lambda_k X_k - 2C_{2m}X_k = (\lambda_k - 2C_{2m})X_k.$$

Thus, $\gamma_k = \lambda_k - 2C_m$ are the eigenvalues of $S_m(n)$. An eigenvalue γ of $S_m(n)$ is a root of the characteristic polynomial $|S_m(n) - \gamma I|$ and this polynomial can be transformed into Chebyshev polynomial of second kind [9], with roots $\gamma_k = -2\cos\frac{\pi k}{n+1}$. Therefore,

$$B_{2m(n+1)} = |T_m(n)| = B_{2m}|R_m(n)| = B_{2m}\prod_{k=1}^{n} \lambda_k = B_{2m}\prod_{k=1}^{n}\left(2C_{2m} - 2\cos\frac{\pi k}{n+1}\right),$$

which follows the factorization (3) by simple change of variables.

References

[1] Behera, A., & Panda, G.K. (1999) On the square roots of triangular numbers, *The Fibonacci Quarterly*, 37(2), 98–105.

[2] Belbachair, H., & Szalay, L. (2014) Balancing in direction $(1, -1)$ in Pascal's triangle, *Armenian Journal of Mathematics*, 6(1), 32–40.

[3] Berczes, A., Liptai, K., & Pink, I. (2010) On generalized balancing numbers, *Fibonacci Quarterly*, 48(2), 121–128.

[4] Keskin R., & Karaatly, O. (2012) Some new properties of balancing numbers and square triangular numbers, *Journal of Integer Sequences*, 15(1).

[5] Liptai, K., Luca, F., Pinter, A., & Szalay, L. (2009) Generalized balancing numbers, *Indagationes Math. N. S.*, 20, 87–100.

[6] Olajos, P. (2010) Properties of balancing, cobalancing and generalized balancing numbers, *Annales Mathematicae et Informaticae*, 37, 125–138.

[7] Panda, G. K., & Ray, P. K. (2011) Some links of balancing and cobalancing numbers with Pell and associated Pell numbers, *Bulletin of the Institute of Mathematics, Academia Sinica (New Series)*, 6(1), 41–72.

[8] Panda, G. K. (2009) Some fascinating properties of balancing numbers, *Proc. Eleventh Internat. Conference on Fibonacci Numbers and Their Applications, Cong. Numerantium*, 194, 185–189.

[9] Ray, P. K. (2012) Application of Chybeshev polynomials in factorization of balancing and Lucas-balancing numbers, *Boletim da Sociedade Paranaense de Matematica*, 30(2), 49–56.

[10] Ray, P. K. (2013) Factorization of negatively subscripted balancing and Lucas-balancing numbers, *Boletim da Sociedade Paranaense de Matematica*, Vol.31 (2), 161–173.

[11] Ray, P. K. (2012) Certain matrices associated with balancing and Lucas-balancing numbers, *Matematika*, 28(1), 15–22.

[12] Ray, P. K. (2013) New identities for the common factors of balancing and Lucas-balancing numbers, *International Journal of Pure and Applied Mathematics*, 85, 487–494.

[13] Ray, P. K. (2014) Some congruences for balancing and Lucas-balancing numbers and their applications, *Integers*, 14, #A8.

[14] Ray, P. K. (2014) Balancing sequences of matrices with application to algebra of balancing numbers, *Notes on Number Theory and Discrete Mathematics*, 20(1), 49–58.

[15] Ray, P. K. (2014) On the properties of Lucas-balancing numbers by matrix method, *Sigmae, Alfenas*, 3(1), 1–6.

[16] Ray, P. K. (2014) Generalization of Cassini formula for balancing and Lucas-balancing numbers, *Matematychni Studii.*, 42(1), 9–14.

[17] Ray, P. K. (2015) Balancing and Lucas-balancing sums by matrix methods, *Mathematical Reports*, 17(67), 2.