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On some Pascal's like triangles. Part 11

Krassimir T. Atanassov

Department of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering – Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 105, Sofia-1113, Bulgaria
e-mail: krat@bas.bg

Abstract: In a series of papers, Pascal's like triangles with different forms have been described. Here, new types of triangles is discussed. In the formula for their generation, operation summation is replaced, respectively, by operations multiplication and exponentiation. Some of their properties are studied. The general case is discussed.
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1 Introduction

In a series of papers [1–7], we discussed a new type of Pascal's Like Triangles (PLTs). Now, we modify the form of operation (in [1–6] it was summation and in [8] – subtraction) with other operations, namely multiplication and exponentiation.

2 Pascal's like triangles with operation multiplication

Let again the elements of the infinite triangles be

$$\begin{array}{ccccccc} & & & a_{1,1} & & & \\ & & a_{2,1} & a_{2,2} & a_{2,3} & & \\ & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} \\ & \vdots & & \vdots & & \vdots & \end{array}$$

where $a_{i,j}$ are arbitrary real (complex) numbers and for every natural number i and:

1. For every natural number j for which $2 \leq j \leq i$ it will be valid:

$$a_{i,j} = a_{i-1,j-1}a_{i,j-1};$$

2. For every natural number j for which $i \leq j \leq 2i-1$ it will be valid:

$$a_{i,j} = a_{i-1,j-1}a_{i,j+1}.$$

Therefore, if we have a generating sequence $\{a_n\}_{n\geq 1}$, then we obtain the PLT

$$\begin{array}{ccccccccc}
 & & & & \mathbf{a_1} & & & & \\
 & & & a_2 & \mathbf{a_1a_2} & a_2 & & & \\
 & & a_3 & a_2a_3 & \mathbf{a_1a_2^2a_3} & a_2a_3 & a_3 & & \\
 & a_4 & a_3a_4 & a_2a_3^2a_4 & \mathbf{a_1a_2^3a_3^3a_4} & a_2a_3^2a_4 & a_3a_4 & a_4 & \\
a_5 & a_4a_5 & a_3a_4^2a_5 & a_2a_3^3a_4^3a_5 & \mathbf{a_1a_2^4a_3^6a_4^4a_5} & a_2a_3^3a_4^3a_5 & a_3a_4^2a_5 & a_4a_5 & a_5 \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Hence, it is valid the following Lemma.

Lemma 1: IF the generating sequence is $\{a_n\}_{n\geq 1}$, then the generated sequence is

$$\left\{ \prod_{i=1}^{n} a_i^{\binom{n-1}{i-1}} \right\}_{n\geq 1}.$$

In the opposite case, if we have the generated sequence $\{a_n\}_{n\geq 1}$, then we obtain the PLT

$$\begin{array}{ccccccccc}
 & & & & \mathbf{a_1} & & & & \\
 & & & \frac{a_2}{a_1} & \mathbf{a_2} & \frac{a_2}{a_1} & & & \\
 & & \frac{a_1a_3}{a_2^2} & \frac{a_3}{a_2} & \mathbf{a_3} & \frac{a_3}{a_2} & \frac{a_1a_3}{a_2^2} & & \\
 & \frac{a_2^3a_4}{a_1a_3^3} & \frac{a_2a_4}{a_3^2} & \frac{a_4}{a_3} & \mathbf{a_4} & \frac{a_4}{a_3} & \frac{a_2a_4}{a_3^2} & \frac{a_2^3a_4}{a_1a_3^3} & \\
\frac{a_1a_3^6a_5}{a_2^4a_4^4} & \frac{a_3^3a_5}{a_2a_4^3} & \frac{a_3a_5}{a_4^2} & \frac{a_5}{a_4} & \mathbf{a_5} & \frac{a_5}{a_4} & \frac{a_3a_5}{a_4^2} & \frac{a_3^3a_5}{a_2a_4^3} & \frac{a_1a_3^6a_5}{a_2^4a_4^4} \\
 & & & \vdots & \vdots & \vdots & & &
\end{array}$$

Therefore, we can prove

Lemma 2: IF the generated sequence is $\{a_n\}_{n\geq 1}$, then the generating sequence is

$$\left\{ \prod_{i=1}^{n} a_i^{(-1)^{n+i}\binom{n-1}{i-1}} \right\}_{n\geq 1}.$$

Now, we give some examples.

$\mathbf{E}_1$:

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 2 & \mathbf{2} & 2 & & & & \\
 & & & \frac{3}{2} & 3 & \mathbf{6} & 3 & \frac{3}{2} & & & \\
 & & \frac{2^3}{3^2} & \frac{2^2}{3} & 4 & \mathbf{24} & 4 & \frac{2^2}{3} & \frac{2^3}{3^2} & & \\
 & \frac{3^3\times 5}{2^7} & \frac{3\times 5}{2^4} & \frac{5}{2^2} & 5 & \mathbf{120} & 5 & \frac{5}{2^2} & \frac{3\times 5}{2^4} & \frac{3^3\times 5}{2^7} & \\
\frac{2^{14}}{3^3\times 5^4} & \frac{2^7}{5^3} & \frac{2^3\times 3}{5^2} & \frac{2\times 3}{5} & 6 & \mathbf{720} & 6 & \frac{2\times 3}{5} & \frac{2^3\times 3}{5^2} & \frac{2^7}{5^3} & \frac{2^{14}}{3^3\times 5^4} \\
 & & & \vdots & & \vdots & & \vdots & & &
\end{array}$$

Obviously, the generated sequence of the above PLT is $\{n!\}_{n\geq 1}$. An **Open problem** is: which is the explicit form of the generating sequence?

When the generated sequence is 2, 3, 5, 7, ..., i.e., the sequence of the prime numbers, then

$\mathbf{E}_2$:

$$\begin{array}{ccccccccc}
 & & & & \mathbf{2} & & & & \\
 & & & \frac{3}{2} & \mathbf{3} & \frac{3}{2} & & & \\
 & & \frac{2\times 5}{3^2} & \frac{5}{3} & \mathbf{5} & \frac{5}{3} & \frac{2\times 5}{3^2} & & \\
 & \frac{3^3\times 7}{2\times 5^3} & \frac{3\times 7}{5^2} & \frac{7}{5} & \mathbf{7} & \frac{7}{5} & \frac{3\times 7}{5^2} & \frac{3^3\times 7}{2\times 5^3} & \\
\frac{2\times 5^6\times 11}{3^4\times 7^4} & \frac{5^3\times 11}{3\times 7^3} & \frac{5\times 11}{7^2} & \frac{11}{7} & \mathbf{11} & \frac{11}{7} & \frac{5\times 11}{7^2} & \frac{5^3\times 11}{3\times 7^3} & \frac{2\times 5^6\times 11}{3^4\times 7^4} \\
 & & & \vdots & & \vdots & & \vdots &
\end{array}$$

If $p_1 = 2, p_2 = 3, p_3 = 5, ...$, then the generating sequence is $\left\{ \prod_{i=1}^{n} p_i^{(-1)^{n+i}\binom{n-1}{i-1}} \right\}_{n\geq 1}$.

We can modify the previous PLT to the form

$$\mathbf{E}_3 : \quad \begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 2 & \mathbf{2} & 2 & & & & \\
 & & & \frac{3}{2} & 3 & \mathbf{6} & 3 & \frac{3}{2} & & & \\
 & & \frac{2\times5}{3^2} & \frac{5}{3} & 5 & \mathbf{30} & 5 & \frac{5}{3} & \frac{2\times5}{3^2} & & \\
 & \frac{3^3\times7}{2\times5^3} & \frac{3\times7}{5^2} & \frac{7}{5} & 7 & \mathbf{210} & 7 & \frac{7}{5} & \frac{3\times7}{5^2} & \frac{3^3\times7}{2\times5^3} & \\
\frac{2\times5^6\times11}{3^4\times7^4} & \frac{5^3\times11}{3\times7^3} & \frac{5\times11}{7^2} & \frac{11}{7} & 11 & \mathbf{2310} & 11 & \frac{11}{7} & \frac{5\times11}{7^2} & \frac{5^3\times11}{3\times7^3} & \frac{2\times5^6\times11}{3^4\times7^4} \\
 & & & \vdots & & \vdots & & \vdots & & &
\end{array}$$

Let i be the imaginary unit. Then we can construct the following PLTs

$$\mathbf{E}_4 : \quad \begin{array}{ccccccccccccc}
 & & & & & & \mathbf{i} & & & & & & \\
 & & & & & i & \mathbf{-1} & i & & & & & \\
 & & & & i & -1 & \mathbf{1} & -1 & i & & & & \\
 & & & i & -1 & 1 & \mathbf{1} & 1 & -1 & i & & & \\
 & & i & -1 & 1 & 1 & \mathbf{1} & 1 & 1 & -1 & i & & \\
 & i & -1 & 1 & 1 & 1 & \mathbf{1} & 1 & 1 & 1 & -1 & i & \\
1 & -1 & 1 & 1 & 1 & 1 & \mathbf{1} & 1 & 1 & 1 & 1 & -1 & i \\
 & & & & \vdots & & \vdots & & \vdots & & & &
\end{array}$$

$$\mathbf{E}_5 : \quad \begin{array}{ccccccccccccc}
 & & & & & & \mathbf{i} & & & & & & \\
 & & & & & 1 & \mathbf{i} & 1 & & & & & \\
 & & & & i & i & \mathbf{-1} & i & i & & & & \\
 & & & 1 & i & -1 & \mathbf{1} & -1 & i & 1 & & & \\
 & & i & i & -1 & 1 & \mathbf{1} & 1 & -1 & i & i & & \\
 & 1 & i & -1 & 1 & 1 & \mathbf{1} & 1 & 1 & -1 & i & 1 & \\
i & i & -1 & 1 & 1 & 1 & \mathbf{1} & 1 & 1 & 1 & -1 & i & i \\
 & & & & \vdots & & \vdots & & \vdots & & & &
\end{array}$$

$\mathbf{E}_6$:

$$
\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{i} & & & & & & \\
 & & & & & -i & \mathbf{1} & -i & & & & & \\
 & & & & i & 1 & \mathbf{1} & 1 & i & & & & \\
 & & & -i & 1 & 1 & \mathbf{1} & 1 & 1 & -i & & & \\
 & & i & 1 & 1 & 1 & \mathbf{1} & 1 & 1 & 1 & i & & \\
 & -i & 1 & 1 & 1 & 1 & \mathbf{1} & 1 & 1 & 1 & 1 & -i & \\
i & 1 & 1 & 1 & 1 & 1 & \mathbf{1} & 1 & 1 & 1 & 1 & 1 & i \\
 & & & & \vdots & & \vdots & & \vdots & & & &
\end{array}
$$

$\mathbf{E}_7$:

$$
\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{i} & & & & & & \\
 & & & & & -1 & -\mathbf{i} & -1 & & & & & \\
 & & & & i & -i & -\mathbf{1} & -i & i & & & & \\
 & & & -1 & -i & -1 & \mathbf{1} & -1 & -i & -1 & & & \\
 & & i & -i & -1 & 1 & \mathbf{1} & 1 & -1 & -i & i & & \\
 & -1 & -i & -1 & 1 & 1 & \mathbf{1} & 1 & 1 & -1 & -i & -1 & \\
i & -i & -1 & 1 & 1 & 1 & \mathbf{1} & 1 & 1 & 1 & -1 & -i & i \\
 & & & & \vdots & & \vdots & & \vdots & & & &
\end{array}
$$

$\mathbf{E}_8$:

$$
\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & i & \mathbf{i} & i & & & & & \\
 & & & & 1 & i & -\mathbf{1} & i & 1 & & & & \\
 & & & i & i & -1 & \mathbf{1} & -1 & i & i & & & \\
 & & 1 & i & -1 & 1 & \mathbf{1} & 1 & -1 & i & 1 & & \\
 & i & i & -1 & 1 & 1 & \mathbf{1} & 1 & 1 & -1 & i & i & \\
1 & i & -1 & 1 & 1 & 1 & \mathbf{1} & 1 & 1 & 1 & -1 & i & 1 \\
 & & & & \vdots & & \vdots & & \vdots & & & &
\end{array}
$$

$\mathbf{E}_9$:

$$
\begin{array}{ccccccccccccc}
 & & & & & & -\mathbf{1} & & & & & & \\
 & & & & & i & -\mathbf{i} & i & & & & & \\
 & & & & -1 & -i & -\mathbf{1} & -i & -1 & & & & \\
 & & & i & -i & -1 & \mathbf{1} & -1 & -i & i & & & \\
 & & -1 & -i & -1 & 1 & \mathbf{1} & 1 & -1 & -i & -1 & & \\
 & i & -i & -1 & 1 & 1 & \mathbf{1} & 1 & 1 & -1 & -i & i & \\
-1 & -i & -1 & 1 & 1 & 1 & \mathbf{1} & 1 & 1 & 1 & -1 & -i & -1 \\
 & & & & \vdots & & \vdots & & \vdots & & & &
\end{array}
$$

$$\mathbf{E}_{10}: \quad \begin{array}{ccccccccccccc}
 & & & & & & \mathbf{i} & & & & & & \\
 & & & & & 1 & \mathbf{i} & 1 & & & & & \\
 & & & & 1 & 1 & \mathbf{i} & 1 & 1 & & & & \\
 & & & 1 & 1 & 1 & \mathbf{i} & 1 & 1 & 1 & & & \\
 & & 1 & 1 & 1 & 1 & \mathbf{i} & 1 & 1 & 1 & 1 & & \\
 & 1 & 1 & 1 & 1 & 1 & \mathbf{i} & 1 & 1 & 1 & 1 & 1 & \\
1 & 1 & 1 & 1 & 1 & 1 & \mathbf{i} & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & \vdots & & \vdots & & \vdots & & & &
\end{array}$$

$$\mathbf{E}_{11}: \quad \begin{array}{ccccccccccccc}
 & & & & & & -\mathbf{i} & & & & & & \\
 & & & & & 1 & -\mathbf{i} & 1 & & & & & \\
 & & & & 1 & 1 & -\mathbf{i} & 1 & 1 & & & & \\
 & & & 1 & 1 & 1 & -\mathbf{i} & 1 & 1 & 1 & & & \\
 & & 1 & 1 & 1 & 1 & -\mathbf{i} & 1 & 1 & 1 & 1 & & \\
 & 1 & 1 & 1 & 1 & 1 & -\mathbf{i} & 1 & 1 & 1 & 1 & 1 & \\
1 & 1 & 1 & 1 & 1 & 1 & -\mathbf{i} & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & \vdots & & \vdots & & \vdots & & & &
\end{array}$$

$$\mathbf{E}_{12}: \quad \begin{array}{ccccccccccccc}
 & & & & & & \mathbf{i} & & & & & & \\
 & & & & & -1 & -\mathbf{i} & -1 & & & & & \\
 & & & & 1 & -1 & \mathbf{i} & -1 & 1 & & & & \\
 & & & 1 & 1 & -1 & -\mathbf{i} & -1 & 1 & 1 & & & \\
 & & 1 & 1 & 1 & -1 & \mathbf{i} & -1 & 1 & 1 & 1 & & \\
 & 1 & 1 & 1 & 1 & -1 & -\mathbf{i} & -1 & 1 & 1 & 1 & 1 & \\
1 & 1 & 1 & 1 & 1 & -1 & -\mathbf{i} & -1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & \vdots & & \vdots & & \vdots & & & &
\end{array}$$

3 Pascal's like triangles with operation exponentiation

In this section we discuss also two different schemes for constructing of Pascal's like triangles.

First, to construct one of the schemes, let the elements of the above infinite triangles be

$$\begin{array}{ccccccc}
 & & & a_{1,1} & & & \\
 & & a_{2,1} & a_{2,2} & a_{2,3} & & \\
 & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & \\
a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} \\
 & \vdots & & \vdots & & \vdots &
\end{array}$$

where $a_{i,j}$ are arbitrary real (complex) numbers and for every natural number i and:
1. for every natural number j for which $2 \leq j \leq i$ it will be valid:

$$a_{i,j} = a_{i-1,j-1}^{a_{i,j-1}};$$

2. for every natural number j for which $i \leq j \leq 2i-1$ it will be valid:

$$a_{i,j} = a_{i-1,j-1}^{a_{i,j+1}}.$$

Second, to construct the other scheme, let us use the same infinite triangle

$$\begin{array}{ccccccc}
 & & & a_{1,1} & & & \\
 & & a_{2,1} & a_{2,2} & a_{2,3} & & \\
 & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & \\
a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} \\
 & \vdots & & \vdots & & \vdots &
\end{array}$$

but here, for every natural number i and:
1. for every natural number j for which $2 \leq j \leq i$ it will be valid:

$$a_{i,j} = a_{i,j-1}^{a_{i-1,j-1}};$$

2. for every natural number j for which $i \leq j \leq 2i-1$ it will be valid:

$$a_{i,j} = a_{i,j+1}^{a_{i-1,j-1}}.$$

Now, for the generating sequence $a, b, c, \ldots$ with elements being real numbers we construct the first type of PLT

$$\mathbf{E}_{13}: \quad \begin{array}{cccccc}
 & & & & \mathbf{a} & \\
 & & & b & \mathbf{b^{a}} & \ldots \\
 & & c & c^{b} & \mathbf{c^{b^{a+1}}} & \ldots \\
 & d & d^{c} & d^{c^{b+1}} & \mathbf{d^{c^{b^{a+1}+b+1}}} & \ldots \\
e \quad e^{d} & e^{d^{c+1}} & & e^{d^{c^{b+1}+c+1}} & \mathbf{e^{d^{c^{b^{a+1}+b+1}+c^{b+1}+c+1}}} & \ldots \\
 & \vdots & & & \vdots & \vdots
\end{array}$$

We can prove by induction the following Lemma.

Lemma 3: If the generating sequence is $\{a_n\}_{n\geq1}$, then the generated sequence is

$$\{a_n^{a_{n-1}^{a_{n-2}^{\cdot^{\cdot^{\cdot^{a_3^{a_2^{a_1+1}+a_2+1}+a_3^{a_2+1}+a_3+1}\cdots a_{n-2}+1}}}}}\}_{n\geq1}.$$

For the generating sequence $a, b, c, ...$ with elements being real numbers we construct the second type of PLT

$$\mathbf{E}_{14}: \begin{array}{cccccc}
 & & & & \mathbf{a} & \\
 & & & b & \mathbf{a^b} & \ldots \\
 & & c & b^c & \mathbf{a^{b^{c+1}}} & \ldots \\
 & d & c^d & b^{c^{d+1}} & \mathbf{a^{b^{c^{d+1}+c+1}}} & \ldots \\
e & d^e & c^{d^{e+1}} & b^{c^{d^{e+1}+d+1}} & \mathbf{a^{b^{c^{d^{e+1}+d+1}+c^{d+1}+c+1}}} & \ldots \\
 & & \vdots & & \vdots & \vdots
\end{array}$$

We can prove by induction the following

Lemma 4: IF the generating sequence is $\{a_n\}_{n\geq1}$, then the generated sequence is

$$\{a_1^{a_2^{a_3^{\cdot^{\cdot^{\cdot^{a_{n-2}^{a_{n-1}^{a_n+1}+a_{n-1}+1}+a_{n-2}^{a_{n-1}+1}+a_{n-2}+1}\cdots a_3+1}}}}}\}_{n\geq1}.$$

4 A general case

Let the generating sequence be $\{a_n\}_{n\geq1}$, where $a_n \in \mathcal{R}$ or $\mathcal{C}$ and $\mathcal{R}$ ($\mathcal{C}$) is the set of the real (complex) numbers. We will discuss two schemas, for both cases of operation subtraction and operation exponentiation. Let function $f : \mathcal{R} \to \mathcal{R}$ ($f : \mathcal{C} \to \mathcal{C}$).

First, let the elements of the infinite triangle with the generating sequence $\{a_n\}_{n\geq 1}$ be

$$\begin{array}{ccccc} & & & a_1 & \\ & & a_2 & f(a_1,a_2) & \dots \\ & a_3 & f(a_2,a_3) & f(f(a_1,a_2),f(a_2,a_3)) & \dots \\ a_4 & f(a_3,a_4) & f(f(a_2,a_3),f(a_3,a_4)) & f(f(f(a_1,a_2),f(a_2,a_3)),f(f(a_2,a_3),f(a_3,a_4))) & \dots \\ & & \vdots & \vdots & \vdots \end{array}$$

It can be easily seen that the n-th element of the generated sequence of the PLT is

$$\underbrace{f...f}_{n-1 \text{ times}}(a_1,a_2)), ..., \ f(f(a_{n-2},a_{n-1}),f(a_{n-1},a_n\underbrace{)...)}_{n-1 \text{ times}}.$$

Second, for the same generating sequence $\{a_n\}_{n\geq 1}$ we obtain the PLT

$$\begin{array}{ccccc} & & & a_1 & \\ & & a_2 & f(a_2,a_1) & \dots \\ & a_3 & f(a_3,a_2) & f(f(a_2,a_1),f(a_3,a_2)) & \dots \\ a_4 & f(a_4,a_3) & f(f(a_3,a_2),f(a_4,a_3)) & f(f(f(a_2,a_1),f(a_3,a_2)),f(f(a_3,a_2),f(a_4,a_3))) & \dots \\ & & \vdots & \vdots & \vdots \end{array}$$

It can be easily seen that the n-th element of the generated sequence of the PLT is

$$\underbrace{f...f}_{n-1 \text{ times}}(a_2,a_1)), ..., \ f(f(a_{n-1},a_{n-2}),f(a_n,a_{n-1}\underbrace{)...)}_{n-1 \text{ times}}.$$

We give an example, using the function ψ, discussed in [**?**]. In it, functions f, g, h coincide with ψ.

For

$$n = \sum_{i=1}^{m} a_i.10^{m-i} \equiv \overline{a_1 a_2 ... a_m},$$

where a_i is a natural number and $0 \leq a_i \leq 9$ $(1 \leq i \leq m)$ let:

$$\varphi(n) = \begin{cases} 0 & \text{, if } n = 0 \\ \\ \sum_{i=1}^{m} a_i & \text{, otherwise} \end{cases}$$

and for the sequence of functions $\varphi_0, \varphi_1, \varphi_2, ...$, where ($l$ is a natural number)

$$\varphi_0(n) = n,$$

$$\varphi_{l+1} = \varphi(\varphi_l(n)),$$

let the function ψ be defined by

$$\psi(n) = \varphi_l(n),$$

in which

$$\varphi_{l+1}(n) = \varphi_l(n).$$

This function has the following (and other) properties:

$$\psi(m+n) = \psi(\psi(m) + \psi(n)),$$

$$\psi(n+9) = \psi(n).$$

Let the sequence $a_1, a_2, ...$ with members – natural numbers, be given, and let

$$c_i = \psi(a_i)\ (i = 1, 2, ...).$$

Hence, we deduce the sequence $c_1, c_2, ...$ from the former sequence. If k and $l \geq 0$ exist, so that

$$c_{i+l} = c_{k+i+l} = c_{2k+i+l} = ...$$

for $1 \leq i \leq k$, then we say that $[c_{l+1}, c_{l+2}, ..., c_{l+k}]$ is a base of the sequence $c_1, c_2, ...$ with length k and with respect to function ψ.

Let the generating sequence be $0, 1, 2, 3, ...$ and let function f be function ψ. Then we obtain the PLT, which only left hand is shown here.

																						0
																					1	1
																				2	3	4
																			3	5	8	3
																		4	7	3	2	5
																	5	9	7	1	3	8
																6	2	2	9	1	4	3
															7	4	6	8	8	9	4	7
														8	6	1	7	6	5	5	9	7
													9	8	5	6	4	1	6	2	2	9
												1	1	9	5	2	6	7	4	6	8	8
											2	3	4	4	9	2	8	6	1	7	6	5
										3	5	8	3	7	7	9	8	5	6	4	1	6
									4	7	3	2	5	3	1	1	9	5	2	6	7	4
								5	9	7	1	3	8	2	3	4	4	9	2	8	6	1
							6	2	2	9	1	4	3	5	8	3	7	7	9	8	5	6
						7	4	6	8	8	9	4	7	3	2	5	3	1	1	9	5	2
					8	6	1	7	6	5	5	9	7	1	3	8	2	3	4	4	9	2
				9	8	5	6	4	1	6	2	2	9	1	4	3	5	8	3	7	7	9
			1	1	9	5	2	6	7	4	6	8	8	9	4	7	3	2	5	3	1	1
		2	3	4	4	9	2	8	6	1	7	6	5	5	9	7	1	3	8	2	3	4
	3	5	8	3	7	7	9	8	5	6	4	1	6	2	2	9	1	4	3	5	8	3
4	7	3	2	5	3	1	1	9	5	2	6	7	4	6	8	8	9	4	7	3	2	5
						⋮					⋮					⋮						

Now, we see that the generated sequence has a base $[1, 4, 3, 5, 8, 3, 7, 7, 9, 8, 5, 6, 4, 1, 6, 2, 2, 9]$ with a length 18 with respect to function ψ. If we denote the elements of this base by $[b_1, b_2, ..., b_{18}]$, then we see that for each i $(1 \leq i \leq 9)$: $b_i + b_{9+i} = 9$.

5 Open problems

An interesting Open problem is: *Find other functions f with interesting properties.*

Obviously, we can give more complex form to the above two schemes, changing the expressions having the form $f(x, y)$ with $f(g(x), h(y))$ and then, the Open problem will be: *Find functions f, g, h with interesting properties.*

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