Notes on Number Theory and Discrete Mathematics
ISSN 1310–5132
Vol. 21, 2015, No. 3, 22–26

A variant of Waring's problem

Abdullah N. Arslan

Department of Computer Science, Texas A&M University
Commerce, TX 75428, USA
e-mail: abdullah.arslan@tamuc.edu

Abstract: We introduce a variant of Waring's problem. For a given positive integer k, consider the problem of writing any given positive integer N as the sum of the kth powers of consecutive integers starting at 1 using each of these kth powers (summands) exactly once, and repeating some of these summands as necessary. Let C_k denote the total number of such repeats. Determine minimum C_k for positive integers k required to write all positive integers using the kth powers of consecutive integers as described. We show that $C_k \leq g(k)$, where $g(k)$ is the usual notation in Waring's problem, the least number of non-negative kth powers sufficient to represent all positive integers. This result implies that for any given positive integer k, every positive integer N can be expressed as a linear combination of the kth powers of consecutive integers with positive integer coefficients that satisfy certain inequalities. Another implication is that for all positive integers N, n', and k, the equation $N = \sum_{i=1}^{n'} i^k x_i$ has at least one solution $(x_1, x_2, \ldots, x_{n'})$ in non-negative integers if $\sum_{i=1}^{n'} i^k \geq N$.
Keywords: Partitions, Lagrange's four square theorem, Hilbert−Waring theorem, function g.
AMS Classification: 11P05, 11Y99, 11D85, 11B83.

1 Introduction

Expressing every integer as sums of squares, cubes, fourth, and larger powers of integers has been a topic of interest for many researchers. Astonishingly this can be done by using a constant number of summands for each power. The four squares conjecture appears in the Arithmetica of Diophantus which was translated into Latin by Bachet in 1621. This conjecture was proven by Lagrange in 1770 and it became Lagrange's four square theorem. Waring's problem asks if every positive integer k has an associated positive integer s such that every positive integer is the sum of at most s kth powers of natural numbers. The Hilbert−Waring theorem, proven by Hilbert in 1909 [4], answers this question affirmatively. For every positive integer k, let $g(k)$

denote the minimum number s of kth powers needed to represent all integers. Clearly, $g(1) = 1$. By Lagrange's four square theorem $g(2) = 4$. It is proven that $g(3) = 9$ [11, 6], $g(4) = 19$ [1, 2], $g(5) = 37$ [5], and $g(6) = 73$ [7]. It has been conjectured that $g(k) = 2k + [(3/2)k] - 2$ for every positive integer k. The sequence of values of $g(k)$ in this expression is known as sequence $A002804$ in the On-Line Encyclopedia of Integer Sequences [9], and the first values are $1, 4, 9, 19, 37, 73, 143, 279, 548, 1079, 2132, 4223, 8384, 16673, 33203, 66190,\ 132055, \ldots$. The Hilbert−Waring theorem [4] implies that $g(k)$ is some constant for all positive integers k.

In Section 2, we introduce a variant of Waring's problem and present an initial result (Theorem 1). Based on this result, in Section 3 we show that 1) for any given positive integer k, every positive integer N can be expressed as a linear combination of the kth powers of consecutive integers with positive integer coefficients that satisfy certain inequalities (Theorem 2); 2) for all positive integers N, n', and k, if $\sum_{i=1}^{n'} i^k \geq N$ then the equation $N = \sum_{i=1}^{n'} i^k x_i$ has at least one solution $(x_1, x_2, \ldots, x_{n'})$ such that for all $i \in [1, n']$, x_i is an integer in $[0, g(k) + 1]$ (Theorem 3). We include our remarks in Section 4, and conclude in Section 5.

2 A variant of Waring's problem and an initial result

We introduce a variant of Waring's problem as the following: Determine the smallest C_k with the property that all positive integers are the sum of a number of kth powers of positive integers, with the set of positive integers consisting of consecutive integers starting at 1, each integer being used once, and at most C_k summands from this set can be re-used (repeated). For example, $N = 456 = 1^2 + 2^2 + 2^2 + 3^2 + 3^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 7^2 + 8^2 + 9^2 + 10^2$, and C_2 can be 3 in this case because each of 2^2 and 7^2 repeats once, 3^2 repeats twice beyond one appearance. We would like to find upper bounds for C_k which are integers that depend on k but not on N if they exist. More formally,

Problem 1 (WMR)**.** *Waring's Problem on Interval of Consecutive Integers with Minimal Repeats (WMR): For any given positive integer k, determine the smallest integer C_k such that all positive integers N can be expressed as the following:*

$$N = \sum_{i=1}^{n} i^k + \sum_{\substack{1 \leq j \leq C_k \\ i_j \in [1, n]}} (i_j)^k, \tag{1}$$

where n is a positive integer.

Theorem 1. *For every positive integer k, $C_k \leq g(k)$, where C_k is the integer described in Problem 1 (WMR).*

Proof. Suppose that N is the integer to be represented, and take n to be the largest integer with the property that the sum s_n of the consecutive kth powers up to n is less than or equal to N, i.e. n is the largest integer such that $s_n = \sum_{i=1}^{n} i^k \leq N$. Then either $N - s_n = 0$ in which case no repeat of any summands is necessary, or by the Hilbert-Waring theorem $N - s_n$ is the sum of at most $g(k)$ kth powers of positive integers all at most n because $N - s_n = N - \sum_{i=1}^{n} i^k < (n+1)^k$. Here $g(k)$ is the usual notation in Waring's problem, the least number of non-negative kth powers sufficient to represent all positive integers. Thus $C_k \leq g(k)$. □

For example, for $N = 1138$, $k = 4$, we see that $n = 5$ since $\sum_{i=1}^{5} i^4 = 979 \leq 1138$, and $\sum_{i=1}^{6} i^4 = 2275 > 1138$. Therefore, $s_n = 979$, and $N - s_n = 159 = 1^4 + 1^4 + 1^4 + 1^4 + 1^4 + 1^4 + 1^4 + 1^4 + 1^4 + 1^4 + 1^4 + 1^4 + 1^4 + 1^4 + 2^4 + 2^4 + 2^4 + 2^4 + 3^4$. In this case, $N = 1138 = s_n + 14.1^4 + 4.2^4 + 3^4$, the number of repeating summands in this case is $19 \leq g(4)$ where $g(4) = 19$ [1, 2].

3 Implications of the initial result

Theorem 1 yields interesting new results that are stated in Theorems 2 and 3.

Theorem 2. *For any positive integer k, every positive integer N can be expressed as $N = \sum_{i=1}^{n} c_i i^k$, where n is a positive integer, for all $i \in [1, n]$, c_i is an integer larger than or equal to 1, and $\sum_{i=1}^{n}(c_i - 1) \leq C_k \leq g(k)$.*

Proof. In Eq. 1 we see that for all i in $[1, n]$, each coefficient $c_i \geq 1$, and each repeat of the kth power of the integer i increases the coefficient c_i by one. Therefore the total number of increments of coefficients is C_k. By Theorem 1, $C_k \leq g(k)$. Thus, $\sum_{i=1}^{n}(c_i - 1) \leq C_k \leq g(k)$. □

Theorem 3. *For all positive integers N, n', and k, if $\sum_{i=1}^{n'} i^k \geq N$ then the equation $N = \sum_{i=1}^{n'} i^k x_i$ has at least one solution $(x_1, x_2, \ldots, x_{n'})$ such that for all $i \in [1, n]$, x_i is an integer in $[0, C_k + 1]$ which is included in $[0, g(k) + 1]$.*

Proof. As in the proof of Theorem 1, let n be the largest integer such that $s_n = \sum_{i=1}^{n} i^k \leq N$. If $\sum_{i=1}^{n'} i^k \geq N$ then $n' \geq n$. We see that $N = \sum_{i=1}^{n'} i^k x_i$ can be partitioned into $\sum_{i=1}^{n} i^k x_i$ and $\sum_{i=n+1}^{n'} i^k x_i$. By Theorems 1 and 2, $N = \sum_{i=1}^{n'} i^k x_i$ has a solution $(x_1^*, x_2^*, \ldots, x_n^*)$, where for all $i \in [1, n]$, x_i^* is an integer, $x_i^* \geq 1$, and $x_i^* \leq C_k + 1 \leq g(k) + 1$ since $\sum_{i=1}^{n}(x_i - 1) \leq C_k \leq g(k)$. By setting $x_{n+1}^* = x_{n+2}^* = \ldots = x_{n'}^* = 0$, it can be easily verified that $(x_1^*, x_2^*, \ldots, x_{n'}^*)$ is a solution to $N = \sum_{i=1}^{n'} i^k x_i$. □

For example, for $n' = 6$, $k = 4$, and $N = 1138$, the equation $1^4 x_1 + 2^4 x_2 + 3^4 x_3 + 4^4 x_4 + 5^4 x_5 + 6^4 x_6 = 1138$ is guaranteed to have a solution in non-negative integers by Theorem 3 because $\sum_{i=1}^{6} i^4 = 2275 \geq 1138$. One such solution is $(x_1, x_2, x_3, x_4, x_5, x_6) = (15, 5, 2, 1, 1, 0)$, where each integer is in $[0, g(4) + 1] = [0, 20]$. We note that without the help of Theorem 3, it is not trivial to see that such an equation has a solution in non-negative integers. In general, given a polynomial $p(x_1, x_2, \ldots, x_n)$ with integer coefficients, determining if $p(x_1, x_2, \ldots, x_n) = 0$ has a solution in natural numbers (Hilbert's tenth problem) is undecidable (not computable) [3]. Clearly, Theorem 3 is applicable only to a restricted set of equations. The problem in this set is computable because all coefficients are positive, and the range for each variable is bounded. However, determining whether a solution exists or not is not immediate. Theorem 3 answers this question and provides a narrow range for each variable if the condition $\sum_{i=1}^{n'} i^k \geq N$ is true.

4 Remarks

Reconsidering the example at the end of Section 2, we see that $N = 1138$ can also be written as $\sum_{i=1}^{4} i^4 + 2^4 + 3.4^4$. This shows that in this particular case the number of repeating summands can be 4 which is strictly smaller than $g(4) = 19$. From this example we also see that choosing the largest n such that $s_n \leq N$ does not always yield a minimal repeat of summands. We pose the following challenge to the research community:

Find optimal C_k's for all, or for particular values of power k which hold for all positive integers N; or, if this is not possible, show that $g(k)$ is the smallest upper bound for C_k.

In an attempt for representing N with minimal repeats of summands using the kth powers of all consecutive integers, in our initial result, we use the maximum integer n such that $s_n \leq N$. One can calculate n that satisfies this from k and N. For example, for $k = 1$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \leq N$. Then $n^2 + n - 2N \leq 0$. The largest integer satisfying this inequality is $n = \lfloor \frac{-1+\sqrt{1+8N}}{2} \rfloor$. For $k = 2$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \leq N$. After finding the real root of the cubic equation $2n^3 + 3n^2 + n - 6N = 0$, we can find the largest integer n such that $2n^3 + 3n^2 + n - 6N \leq 0$. Clearly, $n = \Theta(N^{1/3})$. We conjecture that $n = \Theta(N^{\frac{1}{k+1}})$ for all positive integers k. Formulae for sum of powers of consecutive integers exist for $k = 3, 4, 5, 6$, and general formulae for all k have been conjectured [10]. These formulae, root finding and sign analysis methods for high-degree polynomials can be used for finding n. In our initial result, the existence of n is in our focus. The exact expression for n and computing integers that satisfy the Hilbert-Waring theorem for given N and k are not in our focus. It is sufficient to know that they exist. However, for $k = 2$ we note that one can use randomized polynomial-time algorithms proposed by Rabin and Shallit [8] for finding non-negative integers i_1, i_2, i_3, i_4 such that $N - s_n = m = i_1^2 + i_2^2 + i_3^2 + i_4^2$ in expected running time $O(\log^2 m)$.

5 Conclusion

The Hilbert$-$Waring theorem has a very strong and surprising statement: for all positive integers k, every positive integer N can be expressed as the sum of the kth powers of up to a constant number of integers. We introduce a variant of the Waring's problem with which the objective is to express N as the kth powers of all consecutive integers from an interval, allowing repeats of summands but with the objective of minimizing such repeats. We show that at most $g(k)$ repeats are sufficient (Theorem 1). A natural question after this initial result is if C_k admits upper bounds smaller than $g(k)$, at least for some k's. This calls for further research. Theorem 1 yields two other surprising and interesting results: 1) on expressing every positive integer as a linear combination of powers of consecutive integers with integer coefficients (Theorem 2); and 2) on existence of a solution in non-negative integers to equations with coefficients that are powers of consecutive integers (Theorem 3).

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