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On Π_k–connectivity of some product graphs

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Abstract: Let k be a positive integer. A graph $G = (V, E)$ is said to be Π_k-connected if for any given subset S of $V(G)$ with $|S| = k$, the subgraph induced by S is connected. In this paper, we consider Π_k-connected graphs under different graph valued functions. Π_k-connectivity of Cartesian product, normal product, join and corona of two graphs have been obtained in this paper.
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1 Introduction

Unless mentioned otherwise, for terminology and notation the reader may refer to Harary [3], new ones will be introduced as and when found necessary.

In this article, we consider finite, undirected, simple and connected graphs $G = (V, E)$ with vertex set V and edge set E. As such $p =| V |$ and $q =| E |$ denote the number of vertices and edges of a graph G, respectively. In general, we use $\langle X \rangle$ to denote the sub graph induced by the set of vertices $X \subseteq V$. $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v, respectively. A non-trivial graph G is called connected if any two of its vertices are linked by a path in G.

A Π_k-connected graph G is said to be vertex minimal Π_k-connected if G is not Π_{k-1}-connected. A vertex minimal Π_k-connected graph G is said to be partially vertex–edge minimal Π_k-connected if $G - e$ is not Π_k-connected for atmost $|E(G)| - 1$ edges of G. A vertex minimal Π_k-connected graph G is said to be totally vertex–edge minimal Π_k-connected if $G - e$ is not Π_k-connected for every $e \in E(G)$.

The Cartesian product of two graphs G and H, denoted $G \Box H$, is a graph with vertex set $V(G \Box H) = V(G) \times V(H)$, that is, the set $\{(g, h)/g \in G, h \in H\}$.

The edge set of $G \Box H$ consists of all pairs $[(g_1, h_1), (g_2, h_2)]$ of vertices with $[g_1, g_2] \in E(G)$ and $h_1 = h_2$, or $g_1 = g_2$ and $[h_1, h_2] \in E(H)$.

The normal product of two graphs G and H, denoted $G \bigoplus H$, is a graph with vertex set $V(G \bigoplus H) = V(G) \times V(H)$, that is, the set $\{(g, h)/g \in G, h \in H\}$, and an edge $[(g_1, h_1), (g_2, h_2)]$ exists whenever any of the following conditions hold good:
(i) $[g_1, g_2] \in E(G)$ and $h_1 = h_2$,
(ii) $g_1 = g_2$ and $[h_1, h_2] \in E(H)$,
(iii) $[g_1, g_2] \in E(G)$ and $[h_1, h_2] \in E(H)$. Given a digraph $G_0 = (V_0, E_0)$ and a family of digraphs $\{G_v = (V_v, E_v)\}_{v \in V_0}$ indexed by V_0, the generalized lexicographic product, denoted by $G_0[\{G_v\}_{v \in V_0}]$ is defined as the digraph with vertex set $V = \{(u, v)/ v \in V_0 \; and \; w \in V_v\}$ and arc set $E = \{((v, w), (v', w'))/ (v, v') \in E_0 \; or \; (v = v') \; and \; (w, w') \in E_v\}$.
Join of two graphs is denoted by $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all edges joining V_1 with V_2. The corona $G_1 \circ G_2$was defined by Frucht and Harary [1] as the graph G obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2, and then joining the $i'th$ node of G_1 to every node in the $i'th$ copy of G_2.

2 Preliminary results

Theorem 2.1. *A connected graph G is vertex minimal Π_p-connected if and only if it has at least one cut vertex.*

3 Main results

Proposition 3.1. *Any graph G is Π_k-connected if and only if every subgraph of G having order at least k is Π_k-connected.*

Proof. Let G be any Π_k-connected graph. On contrary, suppose there exists a disconnected vertex induced subgraph H of order at least k. Form a k - vertex subset S of $V(H)$ by taking at least one vertex from at least two components of H. The subgraph induced by S is disconnected, a contradiction.

Conversely, suppose every subgraph of order at least k is Π_k-connected. Hence, every subgraph of order k is connected. Therefore, G is Π_k-connected. □

Theorem 3.1. *Prism of a complete graph K_p is totally vertex–edge minimal Π_{p+1}-connected.*

Proof. Let G be the prism of a complete graph K_p. Let G_1 and G_2 be the copies of K_p in the prism. Let T be an arbitrary set of $p + 1$ vertices, hence T contains at least two adjacent vertices

u, v such that $u \in V(G_1)$ and $v \in V(G_2)$, where $\langle V(G_1)\rangle$ and $\langle V(G_2)\rangle$ are complete graphs. Hence, the subgraph induced by T is connected. Now we prove G is not Π_p-connected and $G - e$ is not Π_{p+1}-connected. Now we shall prove G is not Π_p-connected. Let M be the set of vertices consisting of $p-1$ vertices from $V(G_1)$ and a vertex from $V(G_2)$ non-adjacent with any of the $p-1$ vertices. The subgraph induced by M is disconnected and hence G is not Π_p-connected. Now we prove $G - e$ is not Π_{p+1}-connected. Here two cases arise:

- Case(1): $e \in G_1$ or $e \in G_2$, and
- Case(2): e is an edge between G_1 and G_2.

Case(1): Let $S \subset V(G_2)$ consist of $p-1$ vertices. Let $u,\ v \in V(G_1)$ be such that u is not adjacent to any of the vertices of S and $v \in V(G_1)$ be any vertex. In $G - e(= uv)$, the subgraph induced by $\langle S \cup \{u, v\}\rangle$ is disconnected.

Case(2): Let $e = uf(u)$ be any edge between G_1 and G_2. In $G - e(= uf(u))$, the subgraph induced by $V(G_2) \cup u$ is not connected and hence $G - e(= uf(u))$ is not Π_{p+1}-connected. Hence, G is totally vertex–edge minimal Π_{p+1}-connected. □

Remark 1. *Every totally vertex–edge minimal graph is partially vertex–edge minimal but every partially vertex–edge minimal need not be totally vertex–edge minimal.*

Theorem 3.2. *Prism of partially vertex–edge minimal Π_3-connected graph of order $p \geq 3$ is partially vertex–edge minimal Π_{p+2}-connected.*

Proof. Let G be a partially vertex–edge minimal Π_3-connected graph of order p and H be the prism of G. let G_1 and G_2 be two copies of G in the prism of G. Let the matching be the union of edges $(u, f(u))$ for all u in G_1 and $f(u)$ in G_2, where $f : V(G_1) \longrightarrow V(G_2)$ is a bijection from $V(G_1)$ to $V(G_2)$ such that $f(u)$ is the mirror image of u. Let $T \subset V(H)$ be any subset having $p+2$ vertices. It is clear that T contains at least four vertices $u, v, f(u)$ and $f(v)$ such that u is adjacent to $f(u)$ and v is adjacent to $f(v)$. Here two cases arise:

- Case(1): u is adjacent to v, and
- Case(2): u is not adjacent to v.

Case(1): As G_1 is Π_3-connected, every vertex in $T \cap V(G_1)$ is adjacent to at least one of the two vertices u, v. Similarly, every vertex in $T \cap V(G_2)$ is adjacent to at least one of the two vertices $f(u)$, $f(v)$. Hence, the subgraph induced by the set T is connected.

Case(2): Every vertex in $T \cap V(G_1)$ is a common neighbor of u and v and every vertex in $T \cap V(G_2)$ is a common neighbor of $f(u)$ and $f(v)$. Therefore the subgraph induced by the set T is connected. Hence, the prism of G is Π_{p+2} connected. H is not Π_{p+1} connected, as the graph induced by $\{V(G_1) - u\} \cup \{f(u), f(v)\}$ is disconnected, where $f(u)$ is not adjacent to $f(v)$. Since G_1 is partially vertex–edge minimal Π_3-connected, there exists an edge $e \in E(G_1)$ such that $G_1 - e$ is not Π_3-connected, i.e., there exists three vertices $u,\ v$ and w in $V(G_1 - e)$ such that the subgraph induced by $u,\ v$ and w is disconnected. Let u be a vertex not adjacent to v and w.

Hence, the subgraph of $H - e$ induced by $\{V(G_2) - f(u)\} \cup \{u,\ v,\ w\}$ is disconnected. Hence, $H - e$ is not Π_{p+2}-connected.

Case (a): Suppose $e \in V(G_1)$. Since G_1 is partially vertex–edge minimal Π_3-connected graph, $G_1 - e$ is not Π_3-connected, i.e., there exist three vertices u, v, and w whose induced graph is not connected. The subgraph induced by subset $\{V(G_2) - f(u)\} \cup \{u,\ v,\ w\}$ of $V(H - e)$ is not connected. Hence, $H - e$ is not Π_{p+2}-connected. Similarly we can prove $H - e$ is not Π_{p+2}-connected when $e \in V(G_2)$.

Case (b): Suppose $e = (u,\ f(u))$ is contained in the matching. Since G_2 is partially vertex–edge minimal Π_3-connected graph, there exist a vertex $f(v)$ non-adjacent to $f(u))$. The subgraph of $H - e$ induced by $V(G_1) \cup \{f(u)),\ f(v)\}$ is not connected. Hence, $H - e$ is not Π_{p+2}-connected.

Also H is not Π_{p+1} connected, as the subgraph induced by $\{V(G_1) - u\} \cup \{f(u), f(v)\}$, where $f(u)$ is not adjacent to $f(v)$, is disconnected. □

Example:

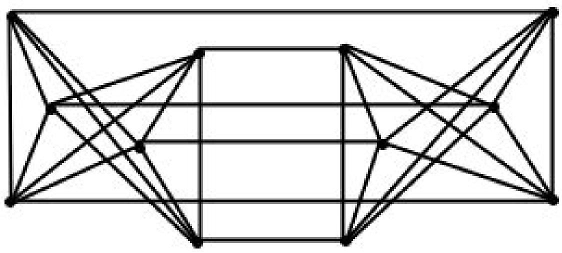


Figure 1: Prism of partially vertex–edge minimal
Π_3-connected graph having even order

Theorem 3.3. *Prism of totally vertex–edge minimal Π_3-connected graph having even order is totally vertex–edge minimal Π_{p+2} connected.*

Proof. Let G be the prism of totally vertex–edge minimal Π_3-connected graph having even order p. G is:

(i) Π_{p+2} connected but not Π_{p+1} connected;

(ii) $G - e_{G_1}$, $G - e_{G_2}$ and $G - e_{G_1,G_2}$ are not Π_{p+2}.

Hence, the prism is totally vertex–edge minimal Π_{p+2} connected. □

Note: In every totally vertex–edge minimal Π_3-connected graph having odd order p, there exist exactly one vertex having degree $p - 1$.

Theorem 3.4. *Prism of totally vertex–edge minimal Π_3-connected graph having odd order p is partially vertex–edge minimal Π_{p+2} connected.*

Proof. Let H be a totally vertex–edge minimal Π_3-connected graph having odd order p and v be a vertex in H having degree $p - 1$. Let G be the prism of H. G is:

(i) Π_{p+2} connected but not Π_{p+1} connected;

(ii) $G - e_{G_1}$ is not Π_{p+2} connected, e is any edge in G_1 not incident with a vertex v, but $G - e_{G_1}$ is still Π_{p+2} connected, if e is incident with v.

Hence, G is partially vertex–edge minimal Π_{p+2} connected. □

The following construction gives the class of non-regular Π_k-connected graphs, where k is a function of p.

Theorem 3.5. *There exist a non-regular Π_{2n+3}-connected graph on $2(2n) + 1,\ n \geq 3$ vertices.*

Proof. Let G be a totally vertex–edge minimal Π_3-connected graph of order $2n$ and $G^{'}$ be a prism of G. Let G_1 be the graph obtained by adding a vertex v and making it adjacent to every vertex of one of the copies in the prism. The resulting graph G_1 is non-regular since $deg(v) = 2n$ and degree of other vertices is $2n - 1$, as v is adjacent to all the vertices in the second copy of the prism. $G_1 - v$ is isomorphic to the graph obtained in theorem(2.3), which is totally vertex–edge minimal Π_{p+2} connected. Let T be any set of $2n + 3$ vertices in G_1. Here two cases arise:

- Case (i): v belongs to T, and
- Case (ii): v does not belong to T.

Case (i): Suppose v belongs to T. The subgraph induced by $T - v$ is connected and contains at least four vertices u, w from the first copy and $f(u), f(w)$ from the second copy such that u is adjacent to $f(u)$ and w is adjacent to $f(w)$ and hence $\langle T \rangle$ is connected as v is adjacent to all the vertices in the second copy.

Case (ii): Suppose v does not belong to T. Clearly the subgraph $\langle T \rangle$ is connected as every Π_k graph is Π_{k+1} also. Hence, G_1 is Π_{2n+3}-connected graph. □

Example:

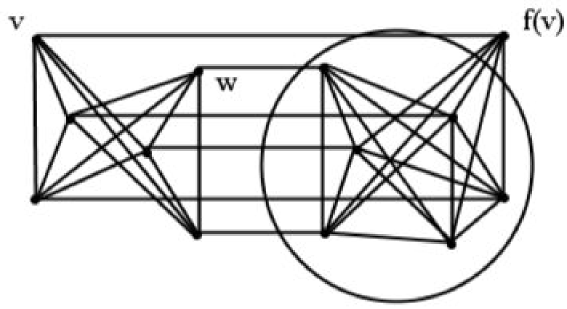


Figure 2: Prism of Π_3-connected graph having even order

Theorem 3.6. *Cartesian product $K_{p_1} \times K_{p_2}$ of two complete graphs K_{p_1} and K_{p_2} is totally vertex–edge minimal Π_k-connected, where $k = p_1p_2 - p_1 - p_2 + 3$.*

Proof. Let V_1 be the set of vertices in K_{p_1} and V_2 be the set of vertices in K_{p_2}. Let $u \in V_1$ and $v \in V_2$. The subgraph induced by the vertices $\{V_1 - u\} \times \{V_2 - v\} \cup (u, v)$ is a disconnected graph. If we include one more vertex, say (u, w) in the above set, the subgraph becomes connected. Hence, the Cartesian product $K_{p_1} \times K_{p_2}$ is vertex minimal Π_k-connected, where $k = p_1p_2 - p_1 - p_2 + 3$ and if we remove $e = (u, w)$, the above subgraph becomes disconnected. Since the vertices and the edge are arbitrarily chosen, the Cartesian product $K_{p_1} \times K_{p_2}$ is totally vertex–edge minimal Π_k-connected, where $k = p_1p_2 - p_1 - p_2 + 3$. Hence, the proof. □

Remark 2. *In the normal product $K_p \bigoplus G$ of a complete graph and any graph G, the subgraph induced by the set of vertices $A \cup B$, where $A = \{(u, x)/\ u \in V(K_p)\}$ and $B = \{(u, y)/\ u \in V(K_p)\}$ is complete bipartite whenever x is adjacent to y in G.*

Remark 3. *Cartesian product of two connected graphs is connected if and only if both are connected.*

Theorem 3.7. *If the graph G is vertex minimal Π_k-connected, $k \geq 3$ then the normal product $K_p \bigoplus G$ is $\Pi_{p(k-1)+1}$-connected.*

Proof. Let a graph G be Π_k-connected. Since G is vertex minimal Π_k-connected, there exist a set (S) of $k - 1$ vertices whose induced subgraph is disconnected and hence the subgraph of $K_p \bigoplus G$ induced by $\langle V(K_p) \times S\rangle$ is disconnected. Any vertex w not in S makes the subgraph induced by $\langle S \cup w\rangle$ connected and hence from the above remark, the subgraph induced by $\langle V(K_p) \times S \ \cup \ (u, w)\rangle$ is connected, where u is any vertex in K_p. Hence, the normal product $K_p \bigoplus G$ is $\Pi_{p(k-1)+1}$-connected. □

Theorem 3.8. *Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are Π_m and Π_n-connected graphs respectively, then $G_1 \bigoplus G_2$ is:*

(i) $\Pi_{p_1(n-1)+1}$*-connected, if* $p_1(n - 1) \geq p_2(m - 1)$;

(ii) $\Pi_{p_2(m-1)+1}$*-connected, if* $p_2(m - 1) > p_1(n - 1)$.

Proof. Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are Π_m and Π_n-connected graphs respectively and $p_1(n - 1) > p_2(m - 1)$. Let $T = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots\}$ be any set of $p_1(n - 1) + 1$ vertices in $G_1 \bigoplus G_2$. Since $p_1(n-1) > p_2(m-1)$, there exists at least m distinct $x_i^{'s}$ and at least n distinct $y_i^{'s}$ in T. Suppose the subgraph induced by T is disconnected, then the subgraphs induced by $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ are disconnected in $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$, a contradiction. Hence, the subgraph induced by T is connected.

Similarly we can prove the second case. □

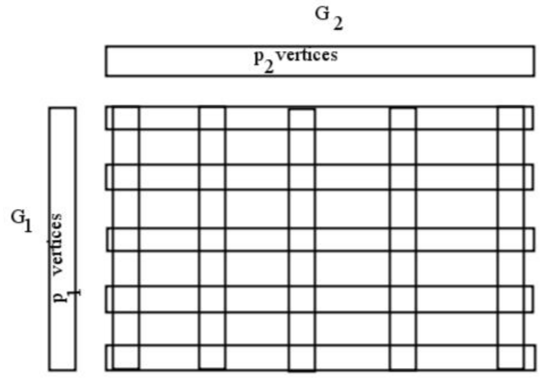


Figure 3: Normal product of two graphs

Remark 4. *Let the graphs $G_1(p, q_1)$ and $G_2(p, q_2)$ are Π_k-connected with $|V(G_1)| < |V(G_2)|$ then $G_1 \bigoplus G_2$ is $\Pi_{|V(G_2)|(k-1)+1}$-connected.*

Theorem 3.9. *Let the graphs G_1 and G_2 are Π_{k_1} and Π_{k_2}-connected with $|V(G_1)| \leq |V(G_2)|$ and $k_1 > k_2$ then $G_1 \bigoplus G_2$ is $\Pi_{|V(G_2)|(k_1-1)+1}$-connected.*

Proof. Let the graphs G_1 and G_2 are Π_{k_1} and Π_{k_2}-connected with $|V(G_1)| \leq |V(G_2)|$ and $k_1 > k_2$. The normal product $G_1 \bigoplus G_2$ is not $\Pi_{|V(G_2)|(k_1-1)}$-connected since G_1 is Π_{k_1}-connected, there exist at least one subset S containing $k_1 - 1$ such that the subgraph $\langle S \rangle$ induced by S is disconnected and hence the subgraph $\langle S \times v(G_2) \rangle$ induced by $S \times v(G_2)$ is disconnected, implies the normal product $G_1 \bigoplus G_2$ is not $\Pi_{|V(G_2)|(k_1-1)}$-connected and $\langle S \times V(G_2) \cup (u, v) \rangle$ is connected, where S and (u, v) are arbitrarily chosen. Hence, $G_1 \bigoplus G_2$ is $\Pi_{|V(G_2)|(k_1-1)+1}$-connected. □

Theorem 3.10. *Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be Π_{k_1} and Π_{k_2}-connected graphs respectively, then the join of G_1 and G_2 is Π_{k_3}-connected, where $k_3 = max\{k_1, k_2\}$.*

Proof. Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be Π_{k_1} and Π_{k_2}-connected graphs respectively. The join is not Π_{k_3-1}-connected because when all the $k_3 - 1$ vertices belongs to $V(G_2)$, where $k_3 = k_2$, then subgraph induced by these $k_3 - 1$ vertices is disconnected. Hence, the join is Π_{k_3}-connected. □

Theorem 3.11. *Let $G_0 = (V_0, E_0)$ be a graph and $\{G_v = (V_v, E_v)\}_{v \in V_0}$ indexed by V_0 be a family of isomorphic graphs then the generalized lexicographic product $G_0[\{G_v\}_{v \in V_0}]$ is Π_t-connected, where $t = (k-1)|V_v| + 1$ if and only if G_0 is Π_k-connected, $k \geq 2$.*

Proof. Let G_0 be a Π_k-connected graph. The lexicographic product $G_0[\{G_v\}_{v \in V_0}]$ is not $\Pi_{(k-1)|V_v|}$-connected as the graph G_0 is not Π_{k-1}-connected, there exist at least one set of $k - 1$ vertices in $V(G_0)$ whose induced subgraph is disconnected in G_0, replacing each of these $k - 1$ vertices by G_v, we get a disconnected $(k-1)|V_v|$ vertex induced subgraph of the generalized lexicographic product $G_0[\{G_v\}_{v \in V_0}]$. The set of $(k-1)|V_v| + 1$ vertices are to be chosen from at least k number of $G_v^{'s}$ and hence the subgraph induced by any $(k-1)|V_v|+1$ vertices is connected as G_0 is Π_k-connected graph.

Conversely, let $G_0[\{G_v\}_{v\in V_0}]$ be Π_t-connected. Suppose on contradiction that G_0 is not Π_k-connected, then there exist at least one set, say S whose cardinality is greater than k. The subgraph, say $G^{'}$ induced after replacing each vertex of S by the set of vertices V_v, is disconnected, i.e., there exists a set of k vertices on which the subgraph induced is not connected and hence by replacing each vertex of S by $V(G_v)$, we get a disconnected subgraph on $(k)|V_v|$, induced in $G_0[\{G_v\}_{v\in V_0}]$, a contradiction to our assumption $G_0[\{G_v\}_{v\in V_0}]$ is Π_t-connected. Hence, G_0 is Π_k-connected. □

Theorem 3.12. *Tensor product of two complete graphs K_{p_1}, $p_1 \geq 3$ and $K_{p_2^{'}}$, $p_2^{'} \geq 2$ is Π_k-connected, where $k = p_1 + p_2^{'}$.*

Proof. We first prove the tensor product of two complete graphs K_{p_1} and K_{p_2} is not Π_{k-1}-connected.

Suppose all the $k-1$ vertices lie on the column and row. Then the vertex lying at the intersection of these column and row is not adjacent to any of the vertices lying in these two row and column and hence the subgraph induced by these $k-1$ vertices is disconnected. Now we prove k is a minimum such that the tensor product is Π_k connected.

Let S be any set of k vertices in the tensor product. Take any vertex v from S and the corresponding row and column containing the vertex v. There exists at least one vertex u not lying on these column and row and hence adjacent to v. Now here we take two cases:

- Case (1) : There exists only one vertex u not lying on these row and column.
- Case (2) : There exists at least two vertices not lying on these row and column.

Case (1): Since the number of vertices in S is $p_1 + p_2'$, there exists at least one vertex x not lying on the row and column containing u. Again the following two sub-cases arise:

(i) x lies on the row or column containing v then x will be adjacent to u and x is also adjacent to all the vertices of S lying on column or row containing v. Hence, the subgraph in this case induced by k vertices is connected.

(ii) x not lying on row or column containing v. Then x will be adjacent to u, also x will be adjacent to at least one vertex in row or column containing v. Hence, in this case also the graph induced by k vertices is connected.

Case (2): There exist at least two vertices not lying on these row and column. Suppose these vertices lie on same row or column then each will be adjacent to at least one vertex lying in column or row as the case may be and hence the subgraph induced by these k vertices is connected. Now suppose these vertices lie on different rows or different columns then each vertex will be adjacent to the other vertices lying in other rows or columns. Hence, in this case also the subgraph induced by k vertices is connected.

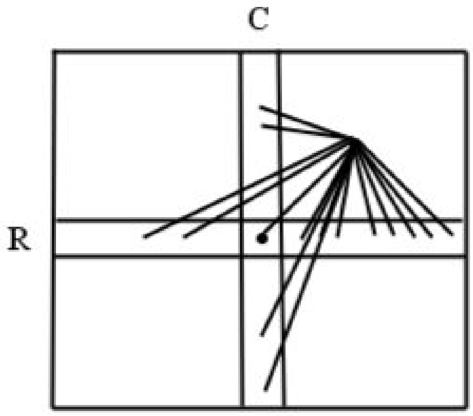
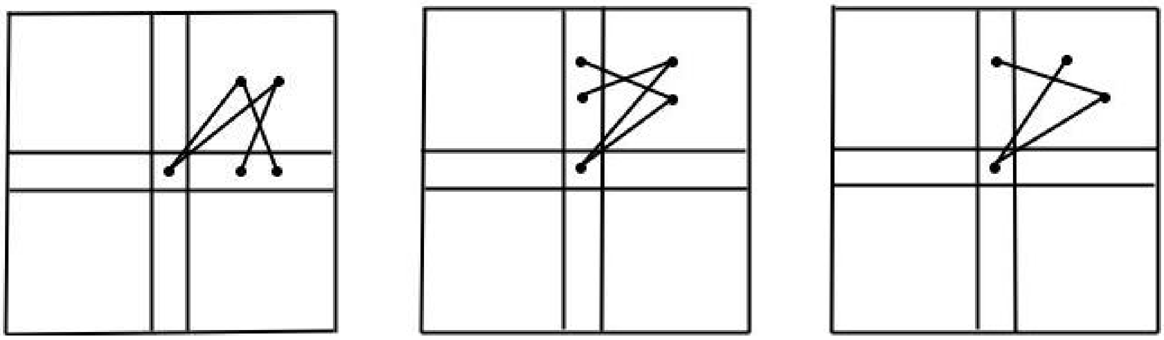


Figure 4: R and C containing $k-1$ vertices in the tensor product



At least two vertices lying in the same row | At least two vertices lying in the same column | Vertices lying in different columns and rows

Figure 5: Graphs for different cases

□

Theorem 3.13. *For any two connected graphs $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$, $G_1 \circ G_2$ is vertex minimal $\Pi_{p_1 p_2}$-connected.*

Proof. Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be any two graphs. Now we prove the $G_1 \circ G_2$ is not $\Pi_{p_1 p_2 - 1}$. The subgraph induced by the vertices $\{u_1, u_2, \ldots, u_{p_1 - 1}\} \cup \{V(G_{u_1}), V(G_{u_2}), \ldots, V(G_{u_{p_1}})\}$ is a disconnected subgraph of $G_1 \circ G_2$. Hence, $G_1 \circ G_2$ is vertex minimal $\Pi_{p_1 p_2}$-connected. □

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