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More new properties of modified Jacobsthal and Jacobsthal–Lucas numbers

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Abstract: We present some new elementary properties of modified Jacobsthal (Atanassov, 2011) and Jacobsthal–Lucas numbers (Shang, 2012).
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1 Introduction

A certain generalization of Jacobsthal numbers in the form

$$J_n^{s,t} = \frac{s^n - (-t)^n}{s+t}, \tag{1}$$

where $n \geq 0$ is a natural number and $s \neq -t$ are arbitrary real numbers was introduced (*see* [2] and [3]). As an analogue, a modification of Jacobsthal–Lucas numbers in the form

$$j_n^{s,t} = s^n + (-t)^n, \tag{2}$$

where n is a natural number and s and t are arbitrary real numbers was proposed [13]. In [10], Rabago studied some elementary properties of these two modifications. For instance, the following relations were obtained in [10]:

$$J_{-n}^{s,t} = (-1)^{n+1} J_n^{s,t}, \quad \forall n \in \mathbb{N}; \tag{3}$$

$$j_{-n}^{s,t} = (-1)^n j_n^{s,t}, \quad \forall n \in \mathbb{N}; \tag{4}$$

$$J_m^{s,t} j_n^{s,t} + j_m^{s,t} J_n^{s,t} = 2J_{m+n}^{s,t} \; ; \tag{5}$$

$$j_m^{s,t} j_n^{s,t} + (s+t)^2 J_m^{s,t} J_n^{s,t} = 2j_{m+n}^{s,t} \; ; \tag{6}$$

$$J_m^{s,t} j_n^{s,t} - j_m^{s,t} J_n^{s,t} = 2(-st)^n J_{m-n}^{s,t}, \quad n < m \; ; \tag{7}$$

$$j_m^{s,t} j_n^{s,t} - (s+t)^2 J_m^{s,t} J_n^{s,t} = 2(-st)^n j_{m-n}^{s,t}, \quad n < m \; ; \tag{8}$$

$$j_m^{s,t} j_n^{s,t} = j_{m+n}^{s,t} + (-st)^n j_{m-n}^{s,t}, \quad n < m \; ; \tag{9}$$

$$J_m^{s,t} j_n^{s,t} = J_{m+n}^{s,t} + (-st)^n J_{m-n}^{s,t}, \quad n < m \; ; \tag{10}$$

$$\left(j_n^{s,t}\right)^2 - (s+t)^2 \left(J_n^{s,t}\right)^2 = 4(-st)^n, \tag{11}$$

where m and n are natural numbers. Also, in [11], Rabago obtained several identities for modified Jacobsthal and Jacobsthal–Lucas numbers using matrix algebra. Recently, Arunkumar, Kannan and Srikanth [1] presented two new properties involving other modifications of Jacobsthal numbers. Particularly, they obtained the following results:

$$(2m+3)JP_s^n = (m+1)\sum_{x=0}^{n-1}\binom{x}{n}2^{n-x}J_{n-x}^m$$

where $2m+3$ and $2m+1$ are both prime numbers and n is any natural number, and

$$(2f_{s+1}+f_s)JF_n^{s+2} > f_{s+1}JF_n^s.$$

Here, JP_s^n and JF_n^s are certain modifications of Jacobsthal numbers as defined by Atanassov in [3]. In this note, we present more results concerning the modifications of Jacobsthal and Jacobsthal–Lucas numbers given by equations (1) and (2).

2 Main results

We start-off in proving the following results using the identities presented in the previous section.

Theorem 2.1. *For every natural number n, we have*

$$J_{2n}^{s,t} = j_n^{s,t} J_n^{s,t}. \tag{12}$$

Proof. The proof is straightforward. Using (1) yields

$$J_{2n}^{s,t} = \frac{s^{n+n} - (-t)^{n+n}}{s+t} = s^n\left(\frac{s^n-(-t)^n}{s+t}\right) + (-t)^n\left(\frac{s^n-(-t)^n}{s+t}\right) = j_n^{s,t} J_n^{s,t}. \tag{13}$$

□

Theorem 2.2. *Let $s \neq -t$ be real numbers. We have, for every natural numbers k and n,*

$$J_{kn}^{s,t} = j_k^{s,t} J_{k(n-1)}^{s,t} - (-st)^k J_{k(n-2)}^{s,t}. \tag{14}$$

Proof. We use equation (5) to prove the theorem, that is,

$$\begin{aligned} J_{kn}^{s,t} &= J_{k+k(n-1)}^{s,t} \\ &= \frac{1}{2}\left(j_k^{s,t} J_{k(n-1)}^{s,t} + J_k^{s,t} j_{k(n-1)}^{s,t}\right) \\ &= \frac{1}{2}\left(j_k^{s,t} J_{k(n-1)}^{s,t} + j_k^{s,t} J_{k(n-1)}^{s,t} - 2(-st)^k J_{k(n-2)}^{s,t}\right) \\ &= j_k^{s,t} J_{k(n-1)}^{s,t} - (-st)^k J_{k(n-2)}^{s,t}, \end{aligned} \tag{15}$$

which is desired. □

Theorem 2.3. *Let s and t be real numbers. We have, for every natural numbers k and n,*

$$j_{kn}^{s,t} = j_k^{s,t} j_{k(n-1)}^{s,t} - (-st)^k j_{k(n-2)}^{s,t}. \tag{16}$$

Proof. We follow the proof of the previous theorem. That is, by using equation (6), we get

$$\begin{aligned} j_{kn}^{s,t} &= j_{k+k(n-1)}^{s,t} \\ &= \frac{1}{2}\left(j_k^{s,t} j_{k(n-1)}^{s,t} + (s+t)^2 J_k^{s,t} J_{k(n-1)}^{s,t}\right) \\ &= \frac{1}{2}\left(j_k^{s,t} j_{k(n-1)}^{s,t} + j_k^{s,t} j_{k(n-1)}^{s,t} - 2(-st)^k j_{k(n-2)}^{s,t}\right) \\ &= j_k^{s,t} j_{k(n-1)}^{s,t} - (-st)^k j_{k(n-2)}^{s,t}. \end{aligned} \tag{17}$$

This proves the theorem. □

Theorem 2.4 (Multiple-angle formulas)**.** *Let $s \neq -t$ be real numbers. We have, for every natural numbers k and n,*

$$J_{kn}^{s,t} = \frac{1}{2^{k-1}} \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2i+1} (s+t)^{2i} \left(J_n^{s,t}\right)^{2i+1} \left(j_n^{s,t}\right)^{k-(2i+1)} \tag{18}$$

$$= \begin{cases} \frac{1}{(s+t)^k} \sum_{i=0}^{k} (-1)^{i+1} \binom{k}{i} J_{k-i}^{s,t} \left(j_n^{s,t}\right)^{k-i} \left(j_{n+1}^{s,t}\right)^{i}, & \textit{for } k \textit{ even};\\ \frac{1}{(s+t)^{k+1}} \sum_{i=0}^{k} (-1)^{i+1} \binom{k}{i} j_{k-i}^{s,t} \left(j_n^{s,t}\right)^{k-i} \left(j_{n+1}^{s,t}\right)^{i}, & \textit{for } k \textit{ odd}. \end{cases} \tag{19}$$

$$= \sum_{i=0}^{k} \binom{k}{i} (st)^{k-i} J_i^{s,t} \left(J_n^{s,t}\right)^{i} \left(J_{n-1}^{s,t}\right)^{k-i}, \ n > 1, \tag{20}$$

$$= \sum_{i=0}^{k} \binom{k}{i} J_{-i}^{s,t} \left(J_n^{s,t}\right)^{i} \left(J_{n+1}^{s,t}\right)^{k-i}. \tag{21}$$

Proof. We let $s \neq -t$ be real numbers and $n, k \in \mathbb{N}$. It can be shown easily that

$$s^n = \frac{j_n^{s,t} + (s+t)J_n^{s,t}}{2}, \quad \forall n \in \mathbb{N} \tag{22}$$

and

$$(-t)^n = \frac{j_n^{s,t} - (s+t)J_n^{s,t}}{2}, \quad \forall n \in \mathbb{N}. \tag{23}$$

Hence,

$$\begin{aligned}
J_{kn}^{s,t} &= \frac{s^{kn} - (-t)^{kn}}{s+t} \\
&= \frac{1}{2^k(s+t)}\left[\left(j_n^{s,t} + (s+t)J_n^{s,t}\right)^k - \left(j_n^{s,t} - (s+t)J_n^{s,t}\right)^k\right] \\
&= \frac{1}{2^k(s+t)}\left\{\binom{k}{0}\left(j_n^{s,t}\right)^k + \binom{k}{1}\left(j_n^{s,t}\right)^{k-1}(s+t)\left(J_n^{s,t}\right) + \cdots + \binom{k}{k}(s+t)^k\left(J_n^{s,t}\right)^k\right. \\
&\quad \left. - \left[\binom{k}{0}\left(j_n^{s,t}\right)^k - \binom{k}{1}\left(j_n^{s,t}\right)^{k-1}(s+t)\left(J_n^{s,t}\right) + \cdots + (-1)^k\binom{k}{k}(s+t)^k\left(J_n^{s,t}\right)^k\right]\right\} \\
&= \frac{1}{2^{k-1}}\sum_{i=0}^{\lfloor (k-1)/2 \rfloor}\binom{k}{2i+1}(s+t)^{2i}\left(J_n^{s,t}\right)^{2i+1}\left(j_n^{s,t}\right)^{k-(2i+1)},
\end{aligned} \tag{24}$$

proving equation (18).

It can also be seen easily that

$$J_n^{s,t} = \frac{st j_{n-1}^{s,t} + j_{n+1}^{s,t}}{(s+t)^2}, \quad \forall n \in \mathbb{N} \tag{25}$$

and

$$j_n^{s,t} = stJ_{n-1}^{s,t} + J_{n+1}^{s,t}, \quad \forall n \in \mathbb{N}. \tag{26}$$

Hence, it is true that

$$s^n = \frac{t j_n^{s,t} + j_{n+1}^{s,t}}{s+t}, \quad \forall n \in \mathbb{N}, \tag{27}$$

and

$$(-t)^n = \frac{s j_n^{s,t} - j_{n+1}^{s,t}}{s+t}, \quad \forall n \in \mathbb{N}. \tag{28}$$

So we have

$$\begin{aligned}
J_{kn}^{s,t} &= \frac{s^{kn} - (-t)^{kn}}{s+t} \\
&= \frac{1}{s+t}\left[\left(\frac{t j_n^{s,t} + j_{n+1}^{s,t}}{s+t}\right)^k - \left(\frac{s j_n^{s,t} - j_{n+1}^{s,t}}{s+t}\right)^k\right] \\
&= \frac{1}{(s+t)^{k+1}}\left(\sum_{i=0}^{k}(-1)^{i+1}\binom{k}{i}\left(s^{k-i} - (-1)^k(-t)^{k-i}\right)\left(j_n^{s,t}\right)^{k-i}\left(j_{n+1}^{s,t}\right)^i\right) \\
&= \begin{cases}
\frac{1}{(s+t)^k}\displaystyle\sum_{i=0}^{k}(-1)^{i+1}\binom{k}{i}J_{k-i}^{s,t}\left(j_n^{s,t}\right)^{k-i}\left(j_{n+1}^{s,t}\right)^i, & \text{for } k \text{ even ;} \\[2ex]
\frac{1}{(s+t)^{k+1}}\displaystyle\sum_{i=0}^{k}(-1)^{i+1}\binom{k}{i}j_{k-i}^{s,t}\left(j_n^{s,t}\right)^{k-i}\left(j_{n+1}^{s,t}\right)^i, & \text{for } k \text{ odd.}
\end{cases}
\end{aligned} \tag{29}$$

On the other hand, it is also true that

$$s^n = sJ_n^{s,t} + stJ_{n-1}^{s,t}, \quad \forall n \in \mathbb{N} \tag{30}$$

and

$$(-t)^n = (-t)J_n^{s,t} + stJ_{n-1}^{s,t}, \quad \forall n \in \mathbb{N}. \tag{31}$$

So we have

$$\begin{aligned} J_{kn}^{s,t} &= \frac{s^{kn} - (-t)^{kn}}{s+t} \\ &= \frac{1}{s+t}\left[\left(sJ_n^{s,t} + stJ_{n-1}^{s,t}\right)^k - \left((-t)J_n^{s,t} + stJ_{n-1}^{s,t}\right)^k\right] \\ &= \sum_{i=0}^{k}\binom{k}{i}(st)^i\left(\frac{s^{k-i} - (-t)^{k-i}}{s+t}\right)\left(J_n^{s,t}\right)^{k-i}\left(J_{n-1}^{s,t}\right)^i \\ &= \sum_{i=0}^{k}\binom{k}{i}(st)^i J_{k-i}^{s,t}\left(J_n^{s,t}\right)^{k-i}\left(J_{n-1}^{s,t}\right)^i, \end{aligned} \tag{32}$$

or equivalently,

$$J_{kn}^{s,t} = \sum_{i=0}^{k}\binom{k}{i}(st)^{k-i}J_i^{s,t}\left(J_n^{s,t}\right)^i\left(J_{n-1}^{s,t}\right)^{k-i}, \ n > 1. \tag{33}$$

Moreover, it can be verified that

$$s^n = J_{n+1}^{s,t} + tJ_n^{s,t}, \quad \forall n \in \mathbb{N} \tag{34}$$

and

$$(-t)^n = J_{n+1}^{s,t} - sJ_n^{s,t}, \quad \forall n \in \mathbb{N}. \tag{35}$$

This yieds

$$\begin{aligned} J_{kn}^{s,t} &= \frac{s^{kn} - (-t)^{kn}}{s+t} \\ &= \frac{1}{s+t}\left(\left(J_{n+1}^{s,t} + tJ_n^{s,t}\right)^k - \left(J_{n+1}^{s,t} - sJ_n^{s,t}\right)^k\right) \\ &= \sum_{i=0}^{k}\binom{k}{i}(-1)^{i+1}\left(\frac{s^i - (-t)^i}{s+t}\right)\left(J_n^{s,t}\right)^i\left(J_{n+1}^{s,t}\right)^{k-i} \\ &= \sum_{i=0}^{k}\binom{k}{i}J_{-i}^{s,t}\left(J_n^{s,t}\right)^i\left(J_{n+1}^{s,t}\right)^{k-i}, \end{aligned} \tag{36}$$

proving equation (21). This completes the proof of the theorem. □

We also have the following theorem for modified Jacobsthal–Lucas numbers.

Theorem 2.5 (Multiple-angle formulas)**.** *Let $s \neq -t$ be real numbers. We have, for every natural numbers k and n,*

$$
\begin{aligned}
j_{kn}^{s,t} &= \frac{1}{2^{k-1}} \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} (s+t)^{2i} \left(J_n^{s,t}\right)^{2i} \left(j_n^{s,t}\right)^{k-2i} && (37)\\
&= \begin{cases} \frac{1}{(s+t)^{k+1}} \sum_{i=0}^{k} (-1)^{i+1} \binom{k}{i} j_{k-i}^{s,t} \left(j_n^{s,t}\right)^{k-i} \left(j_{n+1}^{s,t}\right)^{i}, & \textit{for } k \textit{ even};\\[2ex] \frac{1}{(s+t)^{k}} \sum_{i=0}^{k} (-1)^{i+1} \binom{k}{i} J_{k-i}^{s,t} \left(j_n^{s,t}\right)^{k-i} \left(j_{n+1}^{s,t}\right)^{i}, & \textit{for } k \textit{ odd}. \end{cases} && (38)\\
&= \sum_{i=0}^{k} \binom{k}{i} (st)^{k-i} j_i^{s,t} \left(J_n^{s,t}\right)^{i} \left(J_{n-1}^{s,t}\right)^{k-i}, \ n > 1, && (39)\\
&= \sum_{i=0}^{k} \binom{k}{i} j_{-i}^{s,t} \left(J_n^{s,t}\right)^{i} \left(J_{n+1}^{s,t}\right)^{k-i}. && (40)
\end{aligned}
$$

Proof. The proof follows the same argument as in the previous theorem so we omit it. □

For the following theorems (Theorems 2.6 – 2.10), we shall use an approach similar to Panda and Rout [8] which has been inspired by an earlier result of Behera and Panda [4] on *balancing numbers* (see also [7]).

Theorem 2.6 (Sum of the first n odd indices)**.** *Let $J_n^{s,t}$ denote the n-th modified Jacobsthal number where $s \neq -t$ are real numbers. We have, for all natural number n,*

$$\sum_{i=0}^{n-1} J_{2i+1}^{s,t} = \left(J_n^{s,t}\right)^2 \iff st = -1. \tag{41}$$

Proof. Let $J_n^{s,t}$ denote the n-th modified Jacobsthal number where $s \neq -t$ are real numbers and $n \in \mathbb{N}$. Suppose $\sum_{i=0}^{n-1} J_{2i+1}^{s,t} = (J_n^{s,t})^2$ holds. Hence, using (1), we have

$$
\begin{aligned}
\left(\frac{s^{n-1}-(-t)^{n-1}}{s+t}\right)^2 + \left(\frac{s^{2n-1}-(-t)^{2n-1}}{s+t}\right) &= \left(J_{n-1}^{s,t}\right)^2 + J_{2n-1}^{s,t}\\
&= \sum_{i=0}^{n-2} J_{2i+1}^{s,t} + J_{2n-1}^{s,t}\\
&= \sum_{i=0}^{n-1} J_{2i+1}^{s,t}\\
&= \left(\frac{s^{n}-(-t)^{n}}{s+t}\right)^2.
\end{aligned}
$$

It follows that,

$$\left(\frac{s^{2n-1}-(-t)^{2n-1}}{s+t}\right) = \left(\frac{s^{n}-(-t)^{n}}{s+t}\right)^2 - \left(\frac{s^{n-1}-(-t)^{n-1}}{s+t}\right)^2,$$

or equivalently,

$$(s-(-t))(s^{2n-1}-(-t)^{2n-1})=(s^{2n}-2(-st)^n+t^{2n})-(s^{2n-2}-2(-st)^{n-1}+t^{2n-2}).$$

Expanding the left hand side of the above equation and after some algebra we obtain

$$(-st)(s^{2n-2}+t^{2n-2})=s^{2n-2}+t^{2n-2}+2(-st)^{n-1}(-st-1),$$

which can be further expressed as

$$(-st-1)(s^{n-1}-(-t)^{n-1})^2=(-st-1)(s^{2n-2}-2(-st)^{n-1}+(-t)^{2n-2})=0.$$

Hence, either $st=-1$ or $s^{n-1}=(-t)^{n-1}$. If $s^{n-1}=(-t)^{n-1}$, then $s=\mp t$. By assumption, $s\neq -t$ so $s=t$. Suppose $s=t$, then $J_n^{s,t}=\frac{s^n-(-s)^n}{2s}$. It follows that, for even integer n (i.e. $n=2k, k\in\mathbb{N}$), $J_{2k}^{s,t}=0$, and for odd integer n, $J_{2k-1}^{s,t}=s^{2k-2}$. So $\sum_{k=1}^{n}J_{2k-1}^{s,t}=\sum_{k=1}^{n}(s^2)^{k-1}=\frac{s^{2n}-1}{s^2-1}$. If n is even, then $\frac{s^{2n}-1}{s^2-1}=0$ so $s=t=1$. This implies that, for even integer n, $\sum_{k=1}^{n}J_{2k-1}^{s,t}=\sum_{k=1}^{n}1=n=0=(J_n^{s,t})^2$, a contradiction to our assumption that $n\in\mathbb{N}$. If n is odd, then $\frac{s^{2n}-1}{s^2-1}=(J_n^{s,t})^2=(s^{n-1})^2$ or equivalently, $s^{2n}-1=s^{2n}-s^{2n-2}$. So we have $s=1$ which will lead to a contradiction. We conclude that $st=-1$.

Conversely, if $-st=1$, then we have

$$\begin{aligned}(J_n^{s,t})^2-(J_{n-1}^{s,t})^2 &= \left(\frac{s^n-(-t)^n}{s+t}\right)^2-\left(\frac{s^{n-1}-(-t)^{n-1}}{s+t}\right)^2\\ &= \frac{s^{2n}-2(-st)^n+t^{2n}-(s^{2n-2}-2(-st)^{n-1}+t^{2n-2})}{(s+t)^2}\\ &= \frac{(s^{2n}-(-st)s^{2n-2})+(t^{2n}-(-st)(-t)^{2n-2})}{(s+t)^2}\\ &= \frac{s^{2n-1}(s-(-t))-(-t)^{2n-1}(s-(-t))}{(s+t)^2}\\ &= \frac{s^{2n-1}-(-t)^{2n-1}}{s+t}=J_{2n-1}^{s,t}.\end{aligned}$$

Hence, $(J_n^{s,t})^2-(J_{n-1}^{s,t})^2=J_{2n-1}^{s,t}$. Rearranging the equation and noting that $(J_{n-1}^{s,t})^2=\sum_{i=0}^{n-2}J_{2i+1}^{s,t}$ yields $\sum_{i=0}^{n-1}J_{2i+1}^{s,t}=\sum_{i=0}^{n-2}J_{2i+1}^{s,t}+J_{2n-1}^{s,t}=(J_n^{s,t})^2$. This completes the proof of the theorem. □

Theorem 2.7 (Sum of the first n even indices)**.** *Let $J_n^{s,t}$ denote the n-th modified Jacobsthal number where $s\neq -t$ are real numbers. We have, for all natural number n,*

$$\sum_{i=0}^{n}J_{2i}^{s,t}=J_n^{s,t}J_{n+1}^{s,t}\quad\Longleftrightarrow\quad -st=1. \tag{42}$$

Proof. Let $J_n^{s,t}$ denote the n-th modified Jacobsthal number where $s\neq -t$ are real numbers and $n\in\mathbb{N}$. Note that for any nonzero number $s=t$, $J_n^{s,t}=\frac{s^n-(-s)^n}{s+t}=0$ for all even integer $n\geq 0$. So $\sum_{i=0}^{n-1}J_{2i}^{s,t}=0=J_n^{s,t}J_{n+1}^{s,t}$ is trivially true (because either n or $n+1$ is even). Hence, we may assume (WLOG) that $s \neq t$. The rest follows the proof of the previous theorem. Suppose $\sum_{i=0}^{n} J_{2i}^{s,t} = J_n^{s,t} J_{n+1}^{s,t}$ is true for nonzero real numbers $s \neq \pm t$. Hence, we have

$$J_{n-1}^{s,t} J_n^{s,t} + J_{2n}^{s,t} = \sum_{i=0}^{n-1} J_{2i}^{s,t} + J_{2n}^{s,t} = J_n^{s,t} J_{n+1}^{s,t},$$

which can be expressed as $J_n^{s,t} J_{n+1}^{s,t} - J_{n-1}^{s,t} J_n^{s,t} = J_{2n}^{s,t}$. Using (1), we obtain

$$\begin{aligned} J_n^{s,t}(J_{n+1}^{s,t} - J_{n-1}^{s,t}) &= \frac{s^n - (-t)^n}{s+t} \left(\frac{s^{n+1} - (-t)^{n+1}}{s+t} - \frac{s^{n-1} - (-t)^{n-1}}{s+t} \right) \\ &= \frac{s^{2n+1} + (-t)^{2n+1} - (s^{2n-1} + (-t)^{2n-1})}{(s+t)^2} \\ &\quad - \frac{(-st)^n(s-t) - (-st)^{n-1}(s-t)}{(s+t)^2} \\ &= \frac{s^{2n} - (-t)^{2n}}{s+t} = J_{2n}^{s,t}. \end{aligned}$$

Hence, by rearranging the terms, we get

$$\begin{aligned} (s - (-t))(s^{2n} - (-t)^{2n}) &= s^{2n+1} + (-t)^{2n+1} - (s^{2n-1} + (-t)^{2n-1}) \\ &- (-st)^n(s-t) + (-st)^{n-1}(s-t). \end{aligned}$$

After some algebraic manipulations, we obtain

$$(st+1)[(s^{2n-1} + (-t)^{2n-1}) - (-st)^{n-1}(s-t)] = 0.$$

It follows that, either $-st = 1$ or $(s^{2n-1} + (-t)^{2n-1}) = (-st)^{n-1}(s-t)$. The latter equation is true for all $n \in \mathbb{N}$ provided $s = t$ but, we restrict $s \neq \pm t$, so we conclude that $-st = 1$.

Conversely, suppose that $-st = 1$. Then, it can be verified easily (as in the proof of Theorem (2.6)) that $J_n^{s,t}(J_{n+1}^{s,t} - J_{n-1}^{s,t}) = J_{2n}^{s,t}$. This proves the theorem. □

Note that by using (2.1), we can easily see that, for $s \neq \pm t$, $\sum_{i=0}^{s,t} j_i^{s,t} J_i^{s,t} = J_n^{s,t} J_{n+1}^{s,t}$ if and only if $-st = 1$.

Theorem 2.8. *Let $J_n^{s,t}$ and $j_n^{s,t}$ denote the n-th modified Jacobsthal number and Jacobsthal–Lucas number where $s \neq -t$ are real numbers. We have, for all natural number n,*

$$\left(j_n^{s,t}\right)^2 = (-st)^n + \frac{(s+t)^2}{4} \left(J_n^{s,t}\right)^2. \tag{43}$$

Proof. Let $J_n^{s,t}$ and $j_n^{s,t}$ denote the n-th modified Jacobsthal number and Jacobsthal–Lucas number where $s \neq -t$ are real numbers. Note that

$$(J_n^{s,t})^2 = \left(\frac{s^n - (-t)^n}{s+t} \right)^2 = \frac{s^{2n} + (-t)^{2n} - 2(-st)^n}{(s+t)^2}.$$

Rearranging the equation and doing some algebraic manipulations, we have

$$\frac{(s+t)^2 (J_n^{s,t})^2}{4} + (-st)^n = \frac{s^{2n} + 2(-st)^n + (-t)^{2n}}{4} = \left(\frac{s^n + (-t)^2}{2} \right)^2.$$

Using (2), we can express the above equation as follows

$$\left(j_n^{s,t}\right)^2 = (-st)^n + \frac{(s+t)^2}{4}\left(J_n^{s,t}\right)^2,$$

which is the desired result. □

The following theorem can be veified easily (*see* equation (24) in [11]).

Theorem 2.9. *Let $j_n^{s,t}$ denote the n-th modified Jacobsthal–Lucas number with $-st < 0$ and defined $w_n = j_n^{s,t}/2$. So the sequence $\{w_n\}_{n=1}^{\infty}$ satisfies the recurrence relation $w_{n+1} = (s-t)w_n + stw_{n-1}$ and is an integer sequence if $s-t$ is even with integers s and t.*

Note that $w_0 = j_0^{s,t}/2 = 1$ and $w_1 = j_1^{s,t}/2 = (s-t)/2$ and since w_n satisfies a recurrence relation identical to $J_n^{s,t}$ then w_n is indeed an integer sequence whenever $s-t$ is even. Now, suppose that $-st = 1$ and $s-t = 2l$ for some $l \in \mathbb{N}$. Then, solving for s we obtain $s = l \pm \sqrt{l^2-1}$. If $l = 1$, then we see that $s = 1 = -(-1) = -(-t)$ which is forbidden. So $l > 1$ and this implies that $(s-t)^2 = 4l^2 > 4$ or equivalently, $(s-t)^2 - 4 > 0$. Let $n \in \mathbb{N}$ with $n > 1$ and denote (a, b) as the greatest common divisor of a and b. So $(J_n^{s,t}, w_n) = (J_n^{s,t}, j_n^{s,t}/2) = 1$.

Theorem 2.10. *Let $J_n^{s,t}$ denote the n-th modified Jacobsthal number with $-st = 1$ and $s-t$ be even. We have, for any natural numbers m and n,*

$$n \mid m \iff J_n^{s,t} \mid J_m^{s,t}.$$

Proof. Let $J_n^{s,t}$ denote the n-th modified Jacobsthal number with $-st = 1$ and $s-t$ be even. Suppose $n|m$, i.e. $m = n(k-1)$ for some $k \in \mathbb{N}$. Replacing m by $n(k-1)$ in (5), we obtain

$$\begin{aligned}
(J_n^{s,t}, J_{nk}^{s,t}) &= (J_n^{s,t}, J_{n(k-1)}^{s,t}\frac{j_n^{s,t}}{2} + \frac{j_{n(k-1)}^{s,t}}{2}J_n^{s,t}) \\
&= (J_n^{s,t}, J_{n(k-1)}^{s,t}w_n + w_{n(k-1)}J_n^{s,t}) \\
&= (J_n^{s,t}, J_{n(k-1)}^{s,t})
\end{aligned}$$

Repeatedly applying the same argument, we get $(J_n^{s,t}, J_{nk}^{s,t}) = (J_n^{s,t}, J_n^{s,t}) = J_n^{s,t}$.

Conversely, suppose that $J_n^{s,t}|J_m^{s,t}$. Then, it follows that $n < m$ and by Euclid's algorithm, there exists natural numbers $q \geq 1$ and $0 \leq r < n$ such that $m = nq + r$. Again, using (5),

$$J_n^{s,t} = (J_n^{s,t}, J_m^{s,t}) = (J_n^{s,t}, J_{nq+r}^{s,t}) = (J_n^{s,t}, J_{nq}^{s,t}w_r + w_{nq}J_r^{s,t}).$$

Obviously, n divides nq and so, by our previous result, $J_n^{s,t}|J_{nq}^{s,t}$. It follows that, $J_n^{s,t} = (J_n^{s,t}, w_{nq}J_r^{s,t})$. As we have seen earlier $(J_{nq}^{s,t}, w_{nq}) = 1$ and by iteratively working backwards, we can show that this yields $(J_n^{s,t}, w_{nq}) = 1$. So $J_n^{s,t} = (J_n^{s,t}, J_r^{s,t})$ and this is only possible for $r = 0$ since $0 \leq r < m$ by assumption. Thus, $m = nq$ which concludes that n divides m. Here follows the conclusion. □

We note that Theorem (2.10) still holds for $s = -t$. As we saw earlier, $s = l \pm \sqrt{l^2-1}$ yields $s = 1$ for $l = 1$. It was shown in [11] (*see* equation (52)) that $J_n^{s,-s} = ns^{n-1}$ which is easily obtain by simply letting $s \to -t$ in (1). So for $s = 1$ and $t = -1$, we have $J_n^{1,-1} = n$. Hence, if

m and n are integers and $n|m$, then $J_n^{1,-1}|J_m^{1,-1}$. Obviously, the converse of this statement is also true.

In [6], E. Lucas studied the second-order linear recurrence sequence $\{u_n\}_{n=0}^{\infty}$ defined recursively by $u_{n+2} = Pu_{n+1} - Qu_n$ with initial values $u = 0$ and $u = 1$. He obtained many interesting properties including sums of reciprocals of $\{u_n\}_{n=0}^{\infty}$. For instance, he showed that (*see* equation (125) in [6]), for $k \neq 0$,

$$\sum_{n=1}^{N} \frac{Q^{k2^{n-1}}}{u_{k2^n}} = \frac{Q^k u_{k(2^N-1)}}{u_k u_{k2^N}}. \tag{44}$$

In [11], Rabago showed that, via generating functions, (1) and (2) are the Binet's formulas for the recurrence relations

$$J_{n+1}^{s,t} = (s-t)J_n^{s,t} + stJ_{n-1}^{s,t}, \quad J_0^{s,t} = 0, \quad J_1^{s,t} = 1, \tag{45}$$

and

$$j_{n+1}^{s,t} = (s-t)j_n^{s,t} + stj_{n-1}^{s,t}, \quad j_0^{s,t} = 2, \quad j_1^{s,t} = s-t, \tag{46}$$

respectively (*see* equations (3) and (24) in [11]). He also obtained an analogue of *d'Ocagne's identity* [11]. More precisely, he showed in Theorem 2.16 of [11] that, for $s \neq -t$ and natural numbers m and n such that $n < m$,

$$J_m^{s,t}J_{n+1}^{s,t} - J_n^{s,t}J_{m+1}^{s,t} = (-st)^n J_{m-n}^{s,t}. \tag{47}$$

Equation (47) is an equivalent form of

$$Q^{n-1}u_{m-n} = u_n u_{m-1} - u_m u_{n-1} \tag{48}$$

for the recurrence sequence $\{u_n\}_{n=0}^{\infty}$ studied by Lucas [6]. As pointed out by Rabinowitz in [12], equation (48) can be used to express (44) as follows

$$\sum_{n=1}^{N} \frac{Q^{k2^{n-1}}}{u_{k2^n}} = Q\left[\frac{u_{k(2^N-1)}}{u_{k2^N}} - \frac{u_{k-1}}{u_k}\right]. \tag{49}$$

Lucas [6] also found out that, for $k \neq 0$ and $p \neq 0$,

$$\sum_{n=0}^{N} \frac{Q^{kp^n}u_{k(p-1)p^n}}{u_{kp^n}u_{kp^{n+1}}} = \frac{Q^k u_{k(p^{N+1}-1)}}{u_k u_{kp^{N+1}}}. \tag{50}$$

With these results, we can easily obtained the following theorem.

Theorem 2.11. *Let $J_n^{s,t}$ denote the n-th modified Jacobsthal number where s and t are real numbers such that $s \neq \pm t$. We have, for all $N \in \mathbb{N}$,*

$$\sum_{n=1}^{N} \frac{(-st)^{k2^{n-1}}}{J_{k2^n}^{s,t}} = \frac{(-st)^k J_{k(2^N-1)}^{s,t}}{J_k^{s,t}J_{k2^N}^{s,t}} = (-st)\left[\frac{J_{k(2^N-1)}^{s,t}}{J_{k2^N}^{s,t}} - \frac{J_{k-1}^{s,t}}{J_k^{s,t}}\right]. \tag{51}$$

Popov [9] showed that, for all integers r,

$$\lim_{N\to\infty}\frac{u_{N-r}}{u_N}=\begin{cases}\alpha^r, & \text{if} \quad |\beta/\alpha|<1,\\ \beta^r, & \text{if} \quad |\beta/\alpha|>1.\end{cases}, \tag{52}$$

where α and β are the roots of the quadratic equation $x^2 - Px + Q = 0$. Using these limits, together with Theorem (2.11), we get the following theorem.

Theorem 2.12. *Let $J_n^{s,t}$ denote the n-th modified Jacobsthal number where s and t are real numbers such that $s \neq \pm t$. We have*

$$\sum_{n=1}^{\infty}\frac{(-st)^{k2^{n-1}}}{J_{k2^n}^{s,t}}=\begin{cases}\frac{(-t)^r}{J_k^{s,t}}, & \textit{if} \quad |\beta/\alpha|<1,\\ \\ \frac{s^r}{J_k^{s,t}}, & \textit{if} \quad |\beta/\alpha|>1.\end{cases} \tag{53}$$

References

[1] Arunkumar, S., Kannan, V., & Srikanth, R. (2013) Relations on Jacobsthal numbers, *Notes on Number Theory and Discrete Mathematics*, 19(3), 21–23.

[2] Atanassov, K. T. (2011) Remark on Jacobsthal numbers, Part 2, *Notes on Number Theory and Discrete Mathematics*, 17(2), 37–39.

[3] Atanassov, K. T. (2011) Short remarks on Jacobsthal numbers, *Notes on Number Theory and Discrete Mathematics*, 18(2), 63–64.

[4] Behera, A., & Panda, G. K. (1999) On the square roots of triangular numbers, *The Fibonacci Quaterly*, 37(2), 98–105.

[5] Chandra, P., & Weisstein, E. W. Fibonacci Number. *MathWorld – A Wolfram Web Resource*. Retrieved from `http://mathworld.wolfram.com/FibonacciNumber.html`

[6] Lucas, E. (1878) Théorie des Fonctions Numériques Simplement Périodiques, *American Journal of Mathematics*, 1, 184–240, 289–321; reprinted as *"The Theory of Simply Periodic Numerical Functions"*, Santa Clara, CA: The Fibonacci Association, 1969.

[7] Panda, G. K. (2006) Some fascinating properties of balancing numbers, Applications of Fibonacci Numbers, *Congressus Numerantium*, 194, 185–190.

[8] Panda, G. K., & Rout, S. S. (2012) A Class of Recurrent Sequence Exhibiting Some Exciting Properties of Balancing Numbers, *World Academy of Science, Engineering, and Technology*, 61, 164–166.

[9] Popov, B. S. (1986) Summation of Reciprocal Series of Numerical Functions of Second-Order, *The Fibonacci Quarterly*, 24(1), 17–21.

[10] Rabago, J. F. T. (2013) A note on modified Jacobsthal and Jacobsthal–Lucas numbers, *Notes on Number Theory and Discrete Mathematics*, 19(3), 15–20.

[11] Rabago, J. F. T. (2014) Some new properties of modified Jacobsthal and Jacobsthal–Lucas numbers, *Proceedings of the 3rd International Conference on Mathematical Sciences ICMS3*, AIP Conf. Proc., 1602, 805–818.

[12] Rabinowitz, S. (1998) A Note on the Sum $\sum 1/w_{kw^n}$, *Missouri Journal of Mathematical Sciences*, 10, 141–146.

[13] Shang, Y. (2012) On the modifications of the Pell-Jacobsthal numbers, *Scientia Magna*, 8(3), 68–70.