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Some infinite series involving arithmetic functions

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Abstract: Some identities for infinite series involving arithmetic functions are derived through Jacobi symbols $(-1|k)$ and $(2|k)$. Using these identities, some Dirichlet series are expressed in terms of Hurwitz zeta function.
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1 Introduction

Möbius function arises in many different places in number theory. The Lambert series for Möbius function μ [1, p. 327] is given by

$$\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{1-x^n} = x \quad \text{for } |x| < 1. \tag{1.1}$$

Euler's totient function is an another important arithmetic function that occurs in many identities, special functions or constants. Liouville proved that for $|x| < 1$ [4, p. 182]

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \varphi(n) = \frac{x}{(1-x)^2}. \tag{1.2}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{1+x^n} \varphi(n) = \frac{x(1+x^2)}{(1-x^2)^2}. \tag{1.3}$$

Similar identities for Jordan totient function $J_k(n)$ for $k = 2$ and $|x| < 1$ are given in Ref [4, p. 182] as follows

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} J_2(n) = x\frac{1+x}{(1-x)^3}. \tag{1.4}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{1+x^n} J_2(n) = \frac{x(1+2x+6x^2+2^3+x^4)}{(1-x^2)^3}. \tag{1.5}$$

Cesáro generalized (1.2) for any arithmetic function f

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} f(n) = \sum_{n=1}^{\infty} x^n F(n) \quad \text{for } |x| < 1, \tag{1.6}$$

where $F(n) = \sum_{d|n} f(d)$ [4, p. 182]. In the present study, infinite series involving arithemtic functions are derived through Jacobi symbols $(-1|k)$ and $(2|k)$ for $k \in N$. Further, we derive some identities for Dirichlet series in terms of Hurwitz zeta function.

2 Main theorems

Theorem 2.1. *Let p be an arithmetic function. Let $P(x) = \sum_{k=1,2\nmid k}^{\infty} p(k)(-1|k)x^k$. Then for $|x| < 1$,*

$$P(x) = \sum_{k=1,2\nmid k}^{\infty} (-1|k)a(k)\frac{x^k}{1+x^{2k}}. \tag{2.1}$$

Also, for $Re(s) > 1$

$$\frac{\sum_{k=1}^{\infty}(-1|k)\frac{p(k)}{k^s}}{\sum_{k=1}^{\infty}(-1|k)\frac{a(k)}{k^s}} = \frac{1}{4^s}\left(\zeta(s,1/4) - \zeta(s,3/4)\right), \tag{2.2}$$

where p and a are connected by the following relation

$$a(k) = \sum_{d|k} \mu(d)p(k/d). \tag{2.3}$$

Theorem 2.2. *Let q be an arithmetic function. Let $Q(x) = \sum_{k=1,2\nmid k}^{\infty} q(k)(2|k)x^k$. Then for $|x| < 1$,*

$$Q(x) = \sum_{k=1,2\nmid k}^{\infty} (2|k)\alpha(k)\frac{x^k(1-x^{2k})}{1+x^{4k}}. \tag{2.4}$$

Also, for $Re(s) > 1$

$$\frac{\sum_{k=1}^{\infty}(2|k)\frac{q(k)}{k^s}}{\sum_{k=1}^{\infty}(2|k)\frac{\alpha(k)}{k^s}} = \frac{1}{8^s}\left[\zeta(s,1/8) - \zeta(s,3/8) - \zeta(s,5/8) + \zeta(s,7/8)\right], \tag{2.5}$$

where q and α are connected by the following relation

$$\alpha(k) = \sum_{d|k} \mu(d)q(k/d). \tag{2.6}$$

3 Applications of main theorems

3.1 Infinite series involving arithmetic functions

Let $p(k) = k$ in Theorem 2.1. Then

$$P(x) = \sum_{k=1,2\nmid k}^{\infty} k(-1|k)x^k = \frac{x(1-x^2)}{(1+x^2)^2},$$

and $a(k) = \sum_{d|k} \mu(d)\frac{k}{d} = \varphi(k)$ [1, pp. 26], where φ is Euler totient function. Hence,

$$\frac{x(1-x^2)}{(1+x^2)^2} = \sum_{k=1,2\nmid k}^{\infty} (-1|k)\varphi(k)\frac{x^k}{1+x^{2k}}.$$

Similarly, Theorem 2.2, gives that

$$\frac{x(1-3x^2-3x^4+x^6)}{(1+x^4)^2} = \sum_{k=1,2\nmid k}^{\infty} (2|k)\varphi(k)\frac{x^k(1-x^{2k})}{(1+x^{4k})^2}.$$

Let $f(k) = \left[\frac{z}{k}\right]$ in Theorem 2.1. Then, using the identity $\varphi(z,n) = \sum_{d|n} \mu(d)\left[\frac{z}{d}\right]$, gives

$$\sum_{k=1,2\nmid k} \left[\frac{z}{k}\right](-1|k)x^k = \sum_{k=1,2\nmid k}^{\infty} (-1|k)\varphi(x,k)\frac{x^k}{1+x^{2k}}.$$

Similiarly, using (2.5)

$$\sum_{k=1,2\nmid k} \left[\frac{z}{k}\right](2|k)x^k = \sum_{k=1,2\nmid k}^{\infty} (-1|k)\varphi(x,k)\frac{x^k(1-x^{2k})}{1+x^{4k}}.$$

where $\varphi(z,k)$ is Legendre totient function [4, pp. 283].

Let $f(k) = 1/k$ in Theorem 2.1 and Theorem 2.2. Using the identity

$$b(k) = \frac{1}{k}\sum_{d|k} \mu(d)d = \frac{\varphi^{-1}(k)}{k}$$

[1, pp. 37], gives the following identites

$$\tan^{-1} x = \sum_{k=1,2\nmid k}^{\infty} (-1|k)\varphi^{-1}(k)\frac{x^k}{1+x^{2k}}.$$

$$\int_0^x \frac{1-t^2}{1+t^4}dt = \sum_{k=1,2\nmid k}^{\infty} (2|k)\varphi^{-1}(k)\frac{x^k(1-x^{2k})}{1+x^{4k}},$$

where φ^{-1} is inverse of Euler totient function with respect to convolution.

Consider the identity given in (2.1),

$$P(x) = \sum_{k=1,2\nmid k}^{\infty} (-1|k)a(k)\frac{x^k}{1+x^{2k}}.$$

Dividing above equation by x and integrating on $[0,x]$, yields the following identity

$$\int_0^x \frac{P(t)}{t}dt = \sum_{k=1,2\nmid k}^{\infty} (-1|k)\frac{a(k)}{k}\tan^{-1}(x^k). \tag{3.1}$$

3.2 Evaluation of some Dirichlet series

The following identities are obtained from (2.2) by taking $f(k) = k, 1/k, \log k$ and k^m and respectively.

$$\sum_{k=1}^{\infty} \frac{(-1|k)}{k^{s-1}} = \frac{1}{4^s} \left[\zeta(s, 1/4) - \zeta(s, 3/4)\right] \sum_{k=1}^{\infty} (-1|k) \frac{\varphi(k)}{k^s}.$$

Since $\sum_{k=1}^{\infty} \frac{(-1|k)}{k^{s-1}} = \frac{1}{4^{s-1}} \left[\zeta(s-1, 1/4) - \zeta(s-1, 3/4)\right]$, after simplification gives

$$\sum_{k=1}^{\infty} (-1|k) \frac{\varphi(k)}{k^s} = 4 \frac{\left[\zeta(s-1, 1/4) - \zeta(s-1, 3/4)\right]}{\zeta(s, 1/4) - \zeta(s, 3/4)}.$$

Thus

$$\sum_{k=1}^{\infty} (-1|k) \frac{\varphi^{-1}(k)}{k^s} = \frac{\zeta(s+1, 1/4) - \zeta(s+1, 3/4)}{4\left[\zeta(s, 1/4) - \zeta(s, 3/4)\right]}.$$

$$\sum_{k=1}^{\infty} (-1|k) \frac{J_m(k)}{k^s} = 4^m \frac{\left[\zeta(s-m, 1/4) - \zeta(s-m, 3/4)\right]}{\zeta(s, 1/4) - \zeta(s, 3/4)}.$$

The following identities are obtained from (2.5) by taking $f(k) = k, 1/k, \log k,$ and k^m, respectively.

$$\sum_{k=1}^{\infty} (2|k) \frac{\varphi(k)}{k^s} = 8 \frac{\left[\zeta(s-1, 1/8) - \zeta(s-1, 3/8) - \zeta(s-1, 5/8) + \zeta(s-1, 7/8)\right]}{\left[\zeta(s, 1/8) - \zeta(s, 3/8) - \zeta(s, 5/8) + \zeta(s, 7/8)\right]}.$$

$$\sum_{k=1}^{\infty} (2|k) \frac{\varphi^{-1}(k)}{k^s} = \frac{\left[\zeta(s+1, 1/8) - \zeta(s+1, 3/8) - \zeta(s+1, 5/8) + \zeta(s+1, 7/8)\right]}{8\left[\zeta(s, 1/8) - \zeta(s, 3/8) - \zeta(s, 5/8) + \zeta(s, 7/8)\right]}.$$

$$\sum_{k=1}^{\infty} (2|k) \frac{J_m(k)}{k^s} = 8^m \frac{\left[\zeta(s-m, 1/8) - \zeta(s-m, 3/8) - \zeta(s-m, 5/8) + \zeta(s-m, 7/8)\right]}{\left[\zeta(s, 1/8) - \zeta(s, 3/8) - \zeta(s, 5/8) + \zeta(s, 7/8)\right]}.$$

4 Some lemmas

Lemma 4.1. *For* $|x| < 1$,

$$\sum_{k=1, 2\nmid k}^{\infty} (-1|k) x^k = \frac{x}{1+x^2}. \quad (4.1)$$

$$\sum_{k=1, 2\nmid k}^{\infty} (2|k) x^k = \frac{x(1-x^2)}{1+x^4}. \quad (4.2)$$

Proof. It is well known that for an odd integer k, $(-1|k) = (-1)^{(k-1)/2}$. Then, using binomial theorem, gives (2.1). Similarly, for an odd integer k, $(2|k) = (-1)^{(k^2-1)/8}$. Hence,

$$\sum_{k=1,2\nmid k}^{\infty} (2|k)x^k = x + \sum_{k=0}^{\infty} x^{8k}\left(-x^3 - x^5 + x^7 + x^9\right). \tag{4.3}$$

After simplification, gives (2.2). □

Lemma 4.2. *If* $|x| < 1$*, then*

$$x = \sum_{k=1,2\nmid k}^{\infty} \mu(k)\,(-1|k)\,\frac{x^k}{1+x^{2k}}. \tag{4.4}$$

$$x = \sum_{k=1,2\nmid k}^{\infty} \mu(k)\,(2|k)\,\frac{x^k(1-x^{2k})}{1+x^{4k}}. \tag{4.5}$$

Proof. The left hand side of (2.4) can be written using (2.1) as follows

$$\sum_{k=1,2\nmid k}^{\infty} \mu(k)\,(-1|k)\,\frac{x^k}{1+x^{2k}} = \sum_{k=1,2\nmid k}^{\infty} \mu(k)\,(-1|k) \sum_{m=1,2\nmid m}^{\infty} (-1|m)\,x^{km}.$$

Rearranging above equation, gives

$$\sum_{k=1,2\nmid k}^{\infty} \mu(k)\,(-1|k)\,\frac{x^k}{1+x^{2k}} = \sum_{k=1,2\nmid k}^{\infty} x^k \sum_{d|k} \mu(d)\,(-1|d)\,(-1|k/d).$$

Since $(-1|d)(-1|k/d) = (-1|k)$,

$$\sum_{k=1,2\nmid k}^{\infty} \mu(k)\,(-1|k)\,\frac{x^k}{1+x^{2k}} = \sum_{k=1,2\nmid k}^{\infty} x^k\,(-1|k) \sum_{d|k} \mu(d).$$

Since $\sum_{d|k} \mu(d) = 1$ [3, p. 19] and $(-1|1) = 1$, gives (2.4). Similarly, (2.5) can be easily found from (2.2). □

5 Proof of main theorems

5.1 Proof of Theorem 2.1

Consider

$$P(x) = \sum_{k=1,2\nmid k}^{\infty} (-1|k)p(k)x^k. \tag{5.1}$$

Using (3.4), that

$$P(x) = \sum_{k=1,2\nmid k}^{\infty} (-1|k)p(k) \sum_{m=1,2\nmid m}^{\infty} \mu(m)(-1|m)\frac{x^{mk}}{1+x^{2mk}}.$$

Rearranging the above equation, gives

$$P(x) = \sum_{k=1,2\nmid k}^{\infty} \frac{x^k}{1+x^{2k}} \sum_{d|k} \mu(d)(-1|d)(-1|k/d)p(k/d).$$

Since $(-1|d)(-1|(k/d)) = (-1|k)$. Setting $a(k) = \sum_{d|k} \mu(d)p(k/d)$ in the above equation, gives (2.1). To prove equation (2.2), replace x by e^{-t} in (4.1), multiply by t^{s-1} for $Re(s) > 1$ and integrating on $[0, \infty)$, then

$$\int_0^{\infty} t^{s-1} P(e^{-t}) dt = \sum_{k=1}^{\infty} p(k) \int_0^{\infty} t^{s-1} e^{-kt} dt.$$

Since $\int_0^{\infty} t^{s-1} e^{-kt} dt = \frac{\Gamma(s)}{k^s}$, gives

$$\int_0^{\infty} t^{s-1} P(e^{-t}) dt = \Gamma(s) \sum_{k=1}^{\infty} \frac{p(k)}{k^s}. \tag{5.2}$$

Now, replace x by e^{-t} in (3.1), multiply by t^{s-1} for $Re(s) > 1$ and integrating on $[0, \infty)$, then

$$\int_0^{\infty} t^{s-1} P(e^{-t}) dt = \sum_{k=1}^{\infty} a(k) \int_0^{\infty} \frac{t^{s-1} e^{-kt}}{1+e^{-2kt}} dt$$

After simplification, this gives

$$\int_0^{\infty} t^{s-1} P(e^{-t}) dt = \Gamma(s) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^s} \sum_{k=1}^{\infty} \frac{a(k)}{k^s}.$$

Since $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^s} = \frac{1}{4^s} \left[\zeta(s, 1/4) - \zeta(s, 3/4)\right]$, gives

$$\int_0^{\infty} t^{s-1} P(e^{-t}) dt = \frac{\Gamma(s)}{4^s} \left[\zeta(s, 1/4) - \zeta(s, 3/4)\right] \sum_{k=1}^{\infty} \frac{a(k)}{k^s}. \tag{5.3}$$

Comparing (4.2) and (4.3), gives (2.2). This completes the proof. □

5.2 Proof of Thoerem 2.2

The proof of Theorem 2.2 is similar to proof of Theorem 2.1. Consider

$$Q(x) = \sum_{k=1,2\nmid k}^{\infty} (2|k) q(k) x^k. \tag{5.4}$$

Using (2.5) and Setting $\alpha(k) = \sum_{d|k} \mu(d)q(k/d)$. After simplification, this gives (2.4). Further using the identity

$$\sum_{k=1,2\nmid k}^{\infty} \frac{(2|k)}{k^s} = \frac{1}{8^s} \left[\zeta\left(s, \frac{1}{8}\right) - \zeta\left(s, \frac{3}{8}\right) - \zeta\left(s, \frac{5}{8}\right) + \zeta\left(s, \frac{7}{8}\right)\right]. \tag{5.5}$$

This completes the proof. □

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