Notes on Number Theory and Discrete Mathematics
ISSN 1310–5132
Vol. 21, 2015, No. 1, 67–69

A note on generalized Tribonacci sequence

Aldous Cesar F. Bueno

Department of Mathematics and Physics, Central Luzon State University
Science City of Muñoz 3120, Nueva Ecija, Philippines
e-mail: aldouz_cezar@yahoo.com

Abstract: In this study, we provide a property of the generalized Tribonacci sequence through limits.
Keywords: Tribonacci sequence, Generalized Tribonacci sequence, Tribonacci constant.
AMS Classification: 11B39.

1 Introduction

The Tribonacci sequence $\{T_k\}_{k=0}^{+\infty}$ satisfy the recurrence relation

$$T_n = T_{n-1} + T_{n-2} + T_{n-3},$$

having the initial values $T_0 = 0$, $T_1 = 0$ and $T_2 = 1$. Furthermore, its Binet's formula is given by

$$T_n = \frac{\tau^{n+1}}{(\tau-\sigma)(\tau-\rho)} + \frac{\sigma^{n+1}}{(\sigma-\tau)(\sigma-\rho)} + \frac{\rho^{n+1}}{(\rho-\tau)(\rho-\sigma)},$$

where τ, σ and ρ are the roots of $x^3 - x^2 - x - 1 = 0$. These roots have the following properties:

$$\lim_{n\to+\infty} \frac{\sigma^n}{\tau^n} = 0,$$

$$\lim_{n\to+\infty} \frac{\rho^n}{\tau^n} = 0,$$

and due to these we have,

$$\lim_{n\to+\infty} \frac{T_{n+1}}{T_n} = \tau.$$

Actually, the root τ is called the Tribonacci constant and its explicit form is

$$\tau = \frac{1 + \sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}}}{3}.$$

On the other hand, the generalized Tribonacci sequence denoted by $\{S_k\}_{k=0}^{+\infty}$ satisfy the recurrence relation

$$S_n = S_{n-1} + S_{n-2} + S_{n-3},$$

where the initial values S_1, S_2 and S_3 are arbitrary but are not simultaneously zero.

Natividad and Policarpio [1] provided a formula for finding its n-th term and it is given by

$$S_n = T_{n-2}S_1 + (T_{n-2} + T_{n-3})S_2 + T_{n-1}S_3.$$

Our goal in this study is to investigate the generalized Tribonacci sequence through limits specifically, where we will be dealing on the limit given by

$$\lim_{n\to+\infty} \frac{S_{n+j}}{S_n},$$

where j is a positive integer.

2 Main results

Theorem.

$$\lim_{n\to+\infty} \frac{S_{n+j}}{S_n} = \tau^j.$$

Proof:

$$\begin{aligned}
&\lim_{n\to+\infty} \frac{S_{n+j}}{S_n} \\
&= \lim_{n\to+\infty} \frac{T_{n+j-2}S_1 + (T_{n+j-2} + T_{n+j-3})S_2 + T_{n+j-1}S_3}{T_{n-2}S_1 + (T_{n-2} + T_{n-3})S_2 + T_{n-1}S_3} \\
&= \lim_{n\to+\infty} \frac{T_{n+j-2}S_1/T_n + (T_{n+j-2} + T_{n+j-3})S_2/T_n + T_{n+j-1}S_3/T_n}{T_{n-2}S_1/T-n + (T_{n-2} + T_{n-3})S_2/T_n + T_{n-1}S_3/T_n}.
\end{aligned}$$

Claim:

$$\lim_{n\to+\infty} T_{n+j}/T_n = \tau^j$$

$$\begin{aligned}
&\lim_{n\to+\infty} T_{n+j}/T_n \\
&= \lim_{n\to+\infty} \frac{\frac{\tau^{n+j+1}}{(\tau-\sigma)(\tau-\rho)} + \frac{\sigma^{n+j+1}}{(\sigma-\tau)(\sigma-\rho)} + \frac{\rho^{n+j+1}}{(\rho-\tau)(\rho-\sigma)}}{\frac{\tau^{n+1}}{(\tau-\sigma)(\tau-\rho)} + \frac{\sigma^{n+1}}{(\sigma-\tau)(\sigma-\rho)} + \frac{\rho^{n+1}}{(\rho-\tau)(\rho-\sigma)}} \\
&= \lim_{n\to+\infty} \frac{\frac{\tau^{n+j+1}/\tau^n}{(\tau-\sigma)(\tau-\rho)} + \frac{\sigma^{n+j+1}/\tau^n}{(\sigma-\tau)(\sigma-\rho)} + \frac{\rho^{n+j+1}/\tau^n}{(\rho-\tau)(\rho-\sigma)}}{\frac{\tau^{n+1}/\tau^n}{(\tau-\sigma)(\tau-\rho)} + \frac{\sigma^{n+1}/\tau^n}{(\sigma-\tau)(\sigma-\rho)} + \frac{\rho^{n+1}/\tau^n}{(\rho-\tau)(\rho-\sigma)}} \\
&= \frac{\tau^{j+1}}{\tau};\ by\ the\ properties\ of\ the\ roots\ \tau,\ \sigma,\ and\ \rho \\
&= \tau^j.
\end{aligned}$$

Continuing and using our proven claim, we obtain

$$\begin{aligned}
&\frac{\tau^{j-2}S_1 + (\tau^{j-2} + \tau^{j-3})S_2 + \tau^{j-1}S_3}{\tau^{-2}S_1 + (\tau^{-2} + \tau^{-3})S_2 + \tau^{-1}S_3} \\
&= \frac{\tau^{j+1}S_1 + (\tau^{j+1} + \tau^{j})S_2 + \tau^{j+2}S_3}{\tau S_1 + (\tau + 1)S_2 + \tau^2 S_3} \\
&= \frac{\tau^{j}[\tau S_1 + (\tau + 1)S_2 + \tau^2 S_3]}{\tau S_1 + (\tau + 1)S_2 + \tau^2 S_3} \\
&= \tau^j,
\end{aligned}$$

as desired. □

Corollary.

$$\lim_{n\to+\infty} \frac{S_{n+1}}{S_n} = \tau.$$

3 Conclusion

In summary, we have established that like the Tribonacci sequence, the ratio of the two terms S_{n+j} and S_n of the generalized Tribonacci sequence, given by $\frac{S_{n+j}}{S_n}$, approaches the value τ^j as n tends to infinity where τ is the Tribonacci constant.

References

[1] Natividad, L. R. & Policarpio, P. B. (2013) A novel formula in solving Tribonacci-like sequence, *General Mathematics Notes*, 17(1), 82–87.

[2] Piezas III, T. (2010) A tale of four constants, `https://sites.google.com/site/tpiezas/0012`