Notes on Number Theory and Discrete Mathematics
ISSN 1310–5132
Vol. 21, 2015, No. 1, 42–50

On some Pascal's like triangles. Part 8

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Abstract: In a series of papers, Pascal's like triangles with different forms have been described. Here, three-dimensional analogues of these triangles are given and some of their properties are studied.
Keywords: Pascal pyramid, Pascal triangle, Sequence.
AMS Classification: 11B37.

1 Introduction

In the series of seven papers [1–7], we discussed a new type of Pascal's Like Triangles (PLTs).

Triangles from the present form, but not with the present sense, are described in different publications, e.g. [8–11]. The idea of three-dimensional Pascal object was introduced firstly in [4], where a 4-face pyramid is discussed. Now, we modify this idea for Pascal pyramid with three faces.

It is important to mention that in the first papers of the series we see that there are two basic sequences in a given PLT – the generating and the generated ones, and, it is shown there, we can change their places and roles.

Here, we construct infinite pyramids following the ideas from [1–7]. We show the elements of k-th level as values in a triangle with lenght $(2k + 1)$.

Let us note the following three important lines of the (infinite) pyramid as follows (see Figure 1):

- A-lines – lines that correspond to the three (infinite) pyramid edges,
- B-lines – lines that correspond to the three (infinite) "bisector" faces, and
- C-line – the line that corresponds to the (infinite) pyramid's "altitude".

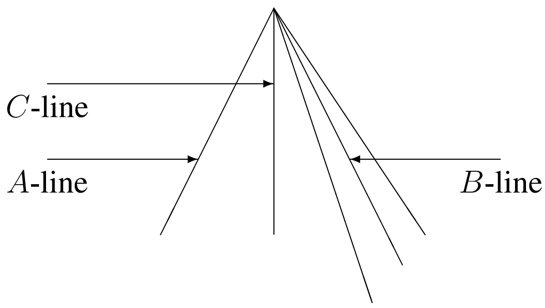

Figure 1. A-, B- and C-lines

In the present research, we discuss only one version of 3-face pyramids, that has the form given in Fig. 1, where there are seven lines:

- three A-lines – A_1-, A_2- and A_3-lines,
- three B-lines – B_1-, B_2- and B_3-lines, and
- one C-line.

Figure 2 is a detailization of Figure 1. In it, the points that correspond to the places of the sequences elements are shown. Now, we can mention that A-lines coincide with the left and right generating sequences and the B-lines – with the generating sequence in the standard PLT.

On the other hand, the C-line corresponds to a generated sequence in a non-standard PTL. The way for obtaining of the values of the members of this sequence will be discussed below.

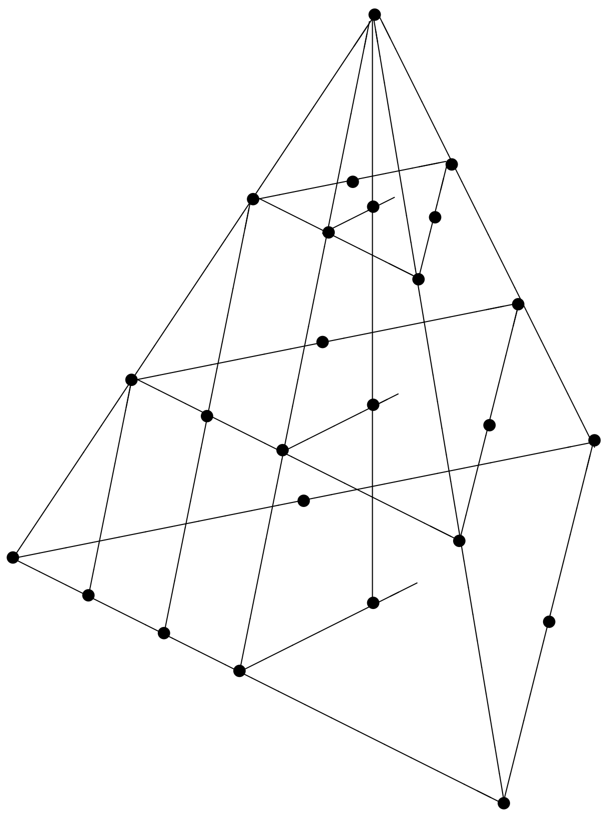

Figure 2.

In Figure 3, we show the first two sequential horisontal sections (levels) of the pyramid, with the points in them, so that each point is symbolized appropriately, keeping, as far as possible, the numeration of the vertices from the previous papers.

Each of the A-, B- and C-lines, that, of course, here represent numerical sequences, has the same first element a. Let the elements of the seven sequences are written in the columns under the line-indices in the following Table 1.

Table 1.

Number of level	A_1	A_2	A_3	B_1	B_2	B_3	C
1	$\alpha_{1,1,1}$	$\alpha_{1,1,1}$	$\alpha_{1,1,1}$	$\alpha_{1,1,1}$	$\alpha_{1,1,1}$	$\alpha_{1,1,1}$	$\alpha_{1,1,1}(=\gamma_1)$
2	$\alpha_{2,1,1}$	$\alpha_{2,1,3}$	$\alpha_{2,3,1}$	$\alpha_{2,1,2}$	$\alpha_{2,2,2}$	$\alpha_{2,2,1}$	γ_2
3	$\alpha_{3,1,1}$	$\alpha_{3,1,5}$	$\alpha_{3,5,1}$	$\alpha_{3,1,3}$	$\alpha_{3,3,2}$	$\alpha_{3,3,1}$	γ_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n\ (n \geq 2)$	$\alpha_{n,1,1}$	$\alpha_{n,1,2n-1}$	$\alpha_{n,2n-1,1}$	$\alpha_{n,1,n}$	$\alpha_{n,n,2}$	$\alpha_{n,n,1}$	γ_n

The α-elements that correspond to points over a pyramid's face are calculated by formulas for a standard PLT from [1].

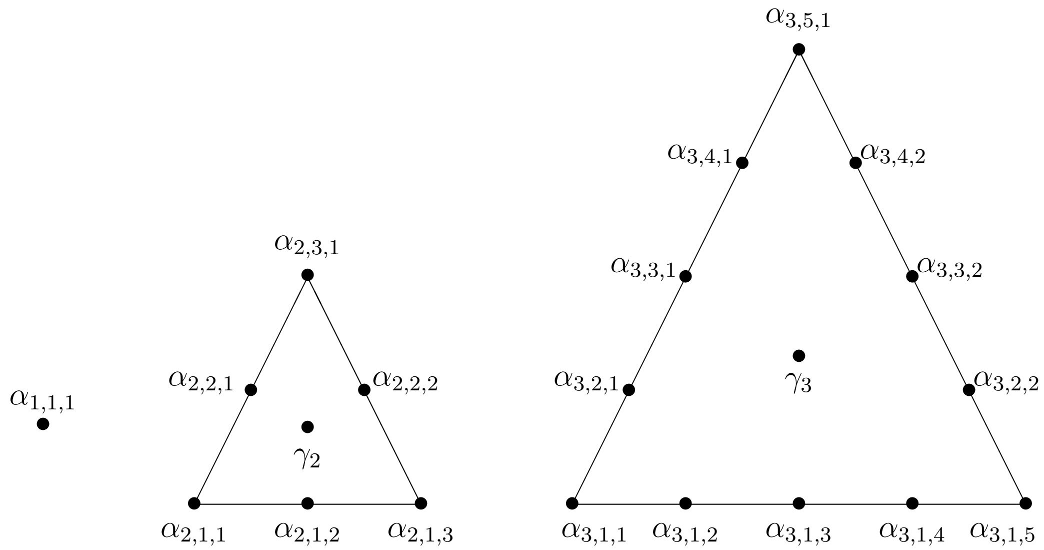

Figure 3.

First, we assume that all the three A-lines coincide. Therefore, all the three B-lines coincide. So, for brevity, when illustrating this case, we omit below the third, fourth, sixth and seventh columns of Table 1. In this case, we use only the triangle generated by one of B-lines and the C-line. So, in this case, the values of the members of the C-line are calculated by formula

$$\gamma_n = \alpha_{n,n,2} + \gamma_{n-1}.$$

Let us start with a pyramid with faces with the form of PLT

1
1 **2** 1
1 2 **4** 2 1
1 2 4 **8** 4 2 1
1 2 4 8 **16** 8 4 2 1
1 2 4 8 16 **32** 16 8 4 2 1
1 2 4 8 16 32 **64** 32 16 8 4 2 1
1 2 4 8 16 32 64 **128** 64 32 16 8 4 2 1
1 2 4 8 16 32 64 128 **256** 128 64 32 16 8 4 2 1
1 2 4 8 16 32 64 128 256 **512** 256 128 64 32 16 8 4 2 1
. . .
. . .

from [1]. It has the form

	A	B	C
1	1	1	1
2	1	2	3
3	1	4	7
4	1	8	15
5	1	16	31
6	1	32	63
7	1	64	127
8	1	128	255
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	1	2^{n-1}	$2^n - 1$

To the pyramid with faces with the form of PLT

1
2 **3** 2
4 6 **9** 6 4
8 12 18 **27** 18 12 8
16 24 36 54 **81** 54 36 24 16
32 48 72 108 162 **243** 162 108 72 48 32
64 96 144 216 324 486 **729** 486 324 216 144 96 64
128 192 288 432 648 972 1458 **2187** 1458 972 648 432 288 192 128
. . .
. . .

the following table corresponds:

	A	B	C
1	1	1	1
2	2	3	4
3	4	9	13
4	8	27	40
5	16	81	121
6	32	243	364
7	64	729	1093
8	128	2187	3280
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	2^{n-1}	3^{n-1}	$\frac{3^n-1}{2}$

For the general case, the pyramid has the form

	A	B	C
1	1	1	1
2	$k-1$	k	$k+1$
3	$(k-1)^2$	k^2	k^2+k+1
4	$(k-1)^3$	k^3	k^3+k^2+k+1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$(k-1)^{n-1}$	k^{n-1}	$\frac{k^n-1}{k-1}$

Below, we give six other examples.

	A	B	C
1	1	1	1
2	0	1	2
3	0	1	3
4	0	1	4
5	0	1	5
6	0	1	6
7	0	1	7
8	0	1	8
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	0	1	n

	A	B	C
1	1	1	1
2	1	2	3
3	0	3	6
4	0	4	10
5	0	5	15
6	0	6	21
7	0	7	28
8	0	8	36
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	0	n	$\frac{n(n+1)}{2}$

	A	B	C
1	1	1	1
2	2	3	4
3	0	5	9
4	0	7	16
5	0	9	25
6	0	11	36
7	0	13	49
8	0	15	64
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	0	$2n-1$	n^2

	A	B	C
1	1	1	1
2	7	8	9
3	12	27	36
4	6	64	100
5	0	125	225
6	0	216	441
7	0	343	784
8	0	512	1296
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	0	n	$\frac{n^2(n+1)^2}{4}$

	A	B	C
1	1	1	1
2	-1	0	1
3	1	1	1
4	-1	0	1
5	1	1	1
6	-1	0	1
7	1	1	1
8	-1	0	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$(-1)^{n-1}$	0	1

	A	B	C
1	0	0	0
2	1	1	1
3	-1	1	2
4	1	1	3
5	-1	1	4
6	1	1	5
7	-1	1	6
8	1	1	7
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$(-1)^n$	1	$n-1$

When the B-line is an arithmetic progression, then the pyramid has the form

	A	B	C
1	a	a	a
2	b	$a+b$	$2a+b$
3	0	$a+2b$	$3a+3b$
4	0	$a+3b$	$4a+6b$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	0	$a+(n-1)b$	$na+\frac{n(n-1)}{2}b$

The 3-dimensional analogue of the PLT

0
1 **1** 1
1 2 **3** 2 1
2 3 5 **8** 5 3 2
3 5 8 13 **21** 13 8 5 3
5 8 13 21 34 **55** 34 21 13 8 5
8 13 21 34 55 89 **144** 89 55 34 21 13 8
13 21 34 55 89 144 233 **377** 233 144 89 55 34 21 13
21 34 55 89 144 233 377 610 **987** 610 377 233 144 89 55 34 ...
. . .
. . .

is the following

	A	B	C
1	0	0	0
2	1	1	1
3	1	3	4
4	2	8	12
5	3	21	33
6	5	55	88
7	8	144	232
8	13	377	609
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	f_n	f_{2n}	$f_{2n+1}-1$

Now, the more complex case of 3-face Pascal's pyramid is discussed. In it, the points are the same, as in Figure 3, but the values of the members of the C-line are calculated by formula

$$\gamma_n = \frac{\alpha_{n,1,1} + \alpha_{n,n,2}}{2} + \gamma_{n-1}.$$

Let the generating sequence that corresponds to the A-lines is $\{2a, 2b, 2c, 2d, ...\}$. Therefore

$$\alpha_{1,1,1} = 2a,$$

$$\alpha_{2,1,1} = \alpha_{2,1,3} = \alpha_{2,3,1} = 2b,$$

$$\alpha_{3,1,1} = \alpha_{3,1,5} = \alpha_{3,5,1} = 2c,$$

$$\alpha_{4,1,1} = \alpha_{4,1,7} = \alpha_{4,7,1} = 2d,$$

$$\alpha_{3,1,1} = \alpha_{3,1,5} = \alpha_{3,5,1} = 2e,$$

$$\dots$$

Then,

$$\alpha_{2,1,2} = \alpha_{2,2,1} = \alpha_{2,2,2} = 2a + 2b,$$

$$\alpha_{3,1,3} = \alpha_{3,3,1} = \alpha_{3,3,2} = 2a + 4b + 2c,$$

$$\alpha_{4,1,4} = \alpha_{4,4,1} = \alpha_{4,4,2} = 2a + 6b + 6c + 2d,$$

$$\dots$$

Hence,

$$\gamma_1 = 2a,$$

$$\gamma_2 = 3a + 2b,$$

$$\gamma_3 = 4a + 5b + 2c,$$

$$\gamma_4 = 5a + 9b + 7c + 2d,$$

$$\gamma_5 = 6a + 14b + 16c + 9d + 2e,$$

$$\gamma_6 = 7a + 20b + 30c + 25d + 11e + 2f,$$

$$\dots$$

By induction, we can prove the following

Lemma. If for some natural number $n \geq 2$ and for a generating sequence $\{x_i\}_{i\geq 1}$:

$$\gamma_n = \sum_{i=1}^{n} \varphi_{n,i} x_i,$$

where $\varphi_{n,1} = n+1$ and $f_{n,n} = 2$, then for every natural number k such that $2 \leq k \leq n-1$, it holds that $\varphi_{n+1,1} = 2n+2,\ \varphi_{n+1,k} = \varphi_{n,k-1} + \varphi_{n,k}, \varphi_{n+1,n+1} = 2.$

The most general form of the 3-face Pascal's pyramid is shown in Figure 4.

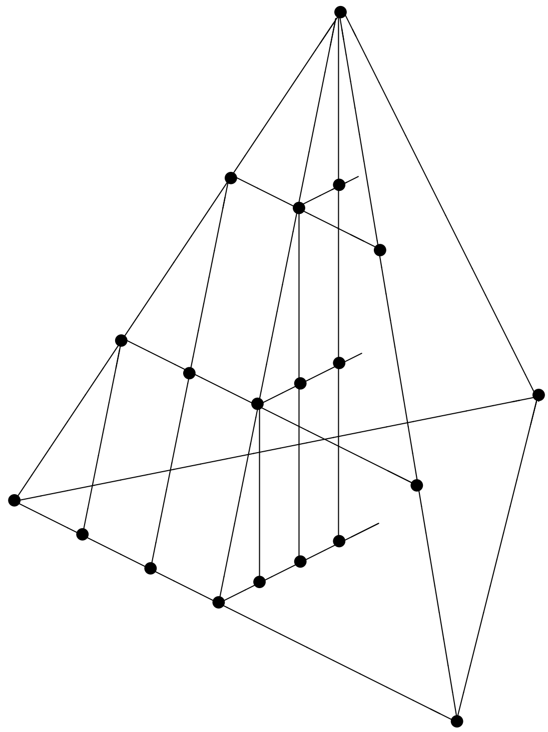

Figure 4.

In addition, in Figure 5, we show the triangles that correspond to the first three levels of the 3-face Pascal's pyramid in Figure 4.

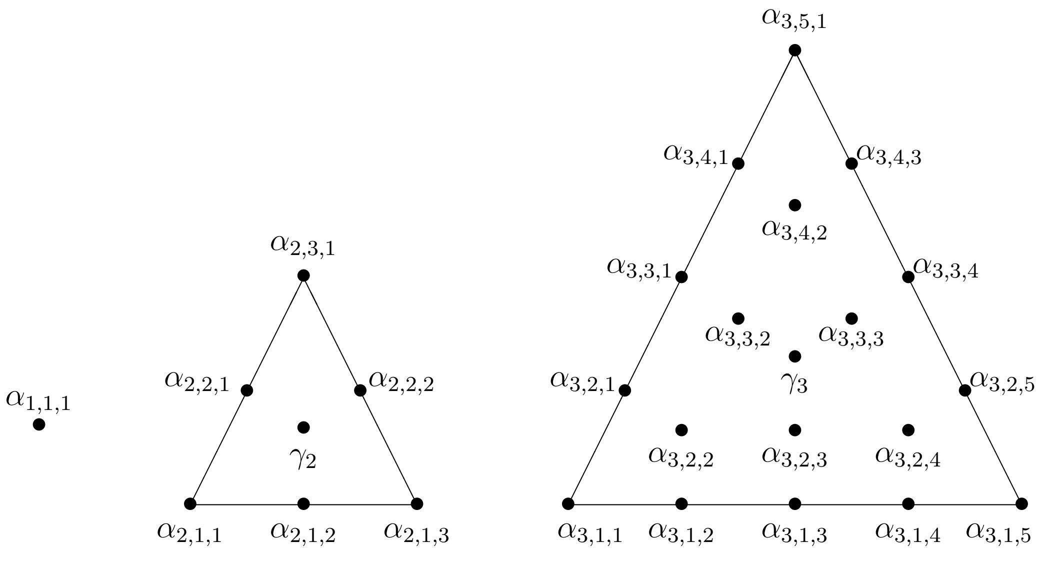

Figure 5.

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