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A basic logarithmic inequality, and the logarithmic mean

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Abstract: By using the basic logarithmic inequality $\ln x \leq x - 1$ we deduce integral inequalities, which particularly imply the inequalities $G < L < A$ for the geometric, logarithmic, resp. arithmetic means.
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1 Introduction

Let $a, b > 0$. The logarithmic mean $L = L(a, b)$ of a and b is defined by

$$L = L(a,b) = \frac{b-a}{\ln b - \ln a} \text{ for } a \neq b \text{ and } L(a,a) = a. \tag{1}$$

Let $G = G(a, b) = \sqrt{ab}$ and $A = A(a, b) = \dfrac{a+b}{2}$ denote the classical geometric, resp. logarithmic means of a and b.

One of the most important inequalities for the logarithmic mean (besides e.g. $a < L(a, b) < b$ for $a < b$) is the following:

$$G < L < A \text{ for } a \neq b \tag{2}$$

The left side of (2) was discovered by B. C. Carlson in 1966 ([1] see [2]), while the right side in 1957 by B. Ostle and H. L. Terwilliger [3].

We note that relation (2) has applications in many subject of pure or applied mathematics and physics including e.g. electrostatics, probability and statistics, etc. (see e.g. [4, 5]).

The following basic logarithmic inequality is well-known:

Theorem 1.

$$\ln x \leq x - 1 \textit{ for all } x > 0. \tag{3}$$

There is equality only for $x = 1$.

Inequality (3) may be proved e.g. by considering the auxiliary function

$$f(x) = x - \ln x - 1,$$

and it is easy to show that $x = 1$ is a global minimum to f, so

$$f(x) \geq f(1) = 0.$$

Another proof is based on the Taylor expansion of the exponential function, yielding $e^t = 1 + t + \frac{t^2}{2} \cdot e^{\theta}$, where $\theta \in (0, t)$. Put $t = x - 1$, and (3) follows.

The continuous arithmetic, geometric and harmonic means of positive, integrable function $f : [a, b] \to \mathbb{R}$ are defined by

$$A_f = \frac{1}{b-a} \int_a^b f(x)dx, \quad G_f = e^{\frac{1}{b-a} \int_a^b \ln f(x)dx}$$

and

$$H_f = \frac{b-a}{\displaystyle\int_a^b dx/f(x)},$$

where $a < b$ are real numbers.

By using (3) we will prove the following classical fact:

Theorem 2.

$$H_f \leq G_f \leq A_f \tag{4}$$

Then, by applying (4) for certain particular functions, we will deduce (2). In fact, (2) will be obtained in a stronger form. The main idea of this note is the use of very simple inequality (3) in the theory of means.

2 The proofs

Proof of Theorem 2. Put

$$x = \frac{(b-a)f(t)}{\displaystyle\int_a^b f(t)dt}$$

in (3), and integrate on $t \in [a, b]$ the obtained inequality. One gets

$$\int_a^b \ln f(t)dt - \left(\left(\frac{1}{b-a} \int_a^b f(t)dt\right)\right)(b-a) \leq \frac{(b-a)\displaystyle\int_a^b f(t)dt}{\displaystyle\int_a^b f(t)dt} - (b-a) = 0.$$

This gives the right side of (4).

Apply now this inequality to $\frac{1}{f}$ in place of f. As

$$\ln \frac{1}{f(t)} = -\ln f(t),$$

we immediately obtain the left side of (4).

Corollary 1. *If f is as above, then*

$$\left(\int_a^b f(t)dt\right)\left(\int_a^b \frac{1}{f(t)}dt\right) \geq (b-a)^2. \tag{5}$$

This follows by $H_f \leq A_f$ in (4).

Remark 1. Let f be continuous in $[a,b]$. The above proof shows that there is equality e.g. in right side of (4) if

$$f(t) = \frac{1}{b-a}\int_a^b f(t)dt. \tag{6}$$

By the first mean value theorem of integrals, there exists $c \in [a,b]$ such that

$$\frac{1}{b-a}\int_a^b f(t)dt = f(c).$$

Since by (6) one has $f(t) = f(c)$ for all $t \in [a,b]$, f is a constant function.

When f is integrable, as

$$\int_a^b \ln\left[(b-a)\frac{f(t)}{\displaystyle\int_a^b f(t)dt}\right]dt = 0,$$

as for $g(t) = \ln \dfrac{(b-a)f(t)}{\displaystyle\int_a^b f(t)dt} > 0$ one has

$$\int_a^b g(t)dt = 0,$$

it follows by a known result that $g(t) = 0$ almost everywhere (a.e.). Therefore

$$f(t) = \frac{1}{b-a}\int_a^b f(t)dt$$

a.e., thus f is a constant a.e.

Remark 2. If f is continuous, it follows in the same manner, that in the left side of (4) there is equality only for $f =$ constant. The same is true for inequality (5).

Proof of (2). Apply $G_f \leq A_f$ to $f(x) = \frac{1}{x}$. Remark that

$$\frac{1}{b-a}\int_a^b \ln x dx = \ln I(a,b),$$

where $a < I(a,b) < b$.

This mean is known in the literature as "identric mean" (see e.g. [4]). As $f(x) = \frac{1}{x}$ is not constant, we get by

$$A_f = \frac{1}{L(a,b)}, \quad G_f = \frac{1}{I(a,b)},$$

that

$$L < I \tag{7}$$

Applying the same inequality $G_f \leq A_f$ to $f(x) = x$ one obtains

$$I < A \tag{8}$$

Remark 3. Inequalities (7) and (8) can be deduced at once by applying all relations of (4) to $f(x) = x$. Apply now (5) to $f(t) = e^t$. After elementary computations, we get

$$\frac{e^b - e^a}{b-a} > e^{\frac{a+b}{2}} \tag{9}$$

As $f(t) > 0$ for any $t \in \mathbb{R}$, inequality (9) holds true for any $a, b \in \mathbb{R}$, $b > a$. Replace now $b := \ln b$, $a := \ln a$, where now the new values of a and b are > 0. One gets from (9):

$$L > G \tag{10}$$

By taking into account of $(7) - -(10)$, we can write:

$$G < L < I < A, \tag{11}$$

i.e. (2) is proved (in improved form on the right side).

Remark 4. Inequality (4) (thus, relation (10)) follows also by $G_f \leq A_f$ applied to $f(t) = e^t$.

Remark 5. The right side of (2) follows also from (5) by the application $f(t) = t$. As

$$\int_a^b t dt = \frac{b^2 - a^2}{2} \text{ and } \int_a^b \frac{1}{t} dt = (\ln b - \ln a),$$

the relation follows.

Remark 6. Clearly, in the same manner as (4), the discrete inequality of means can be proved, by letting $x = \frac{nx_i}{x_1 + \ldots + x_n}$ $(x_1, \ldots, x_n > 0)$.

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