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A note on the Lebesgue–Radon–Nikodym theorem with respect to weighted and twisted p-adic invariant integral on $\mathbb{Z}_p$

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Abstract: In this paper we will give the Lebesgue–Radon–Nikodym theorem with respect to weighted and twisted p-adic q-measure on $\mathbb{Z}_p$. In special case, if there is no twisted, then we can derive the same result as Jeong and Rim, 2012; If the case weight zero and no twist, then we derive the same result as Kim 2012.

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1 Introduction

Let p be a fixed odd prime number. Throughout this paper, the symbols $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|.|_p$ is defined by $|x|_p = p^{-v_p(x)} = p^{-r}$ for $x = p^r \frac{s}{t}$, where s and t are integers with $(p, s) = (p, t) = 1$ and $r \in \mathbb{Q}$ (see [1–16]).

When one speaks of q-extension, q can be regard as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. In this paper we assume that $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ and we use the notations of q-numbers as follows:

$$[x]_q = [x : q] = \frac{1-q^x}{1-q}, \quad \text{and} \quad [x]_{-q} = \frac{1-(-q)^x}{1+q}.$$

For $n \in \mathbb{Z}_+$, let $C_{p^n} = \{w \mid w^{p^n} = 1\}$ be the cyclic group of order p^n and let T_p be the space of locally constant space, i.e, $T_p = \lim_{n\to\infty} C_{p^n} = \cup_{n\geq 0} C_{p^n}$.
For any positive integer N, let

$$a + p^N\mathbb{Z}_p = \left\{x \in \mathbb{Z}_p | x \equiv a \pmod{p^N}\right\} \tag{1}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < p^N$(see [1, 2, 3, 5-9]).

It is known that the fermionic p-adic q-measure on $\mathbb{Z}_p$ is given by Kim as follows:

$$\mu_{-q}(a + p^N\mathbb{Z}_p) = \frac{(-q)^a}{[p^N]_{-q}} = \frac{1+q}{1+q^{p^N}}(-q)^a, \tag{2}$$

(see [5, 7, 12–16]).

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. From (2), the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \tag{3}$$

where $f \in C(\mathbb{Z}_p)$(see [1, 5, 7, 12-16]).

For $w \in T_p$, we consider the twisted q-Euler polynomials $\varepsilon_{n,q}^w(x)$ as

$$\int_{\mathbb{Z}_p} q^{-y} w^y e^{[x+y]_q} d\mu_{-q}(y) = \sum_{n=0}^{\infty} \varepsilon_{n,q}^w(x) \frac{t^n}{n!} \tag{4}$$

From (4), we can derive the following equation.

$$\varepsilon_{n,q}^w(x) = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} (-1)^l \frac{q^{lx}}{1+wq^l}, \tag{5}$$

(see [16]).

From (5), we note that $\lim_{q\to 1} \varepsilon_{n,q}^w(x) = \varepsilon_n^w(x)$.

In special case, $x = 0$, we have $\varepsilon_{n,q}^w(0) = \varepsilon_{n,q}^w$ are the n-th twisted q-Euler number, we have

$$\varepsilon_{n,q}^w = \int_{\mathbb{Z}_p} q^{-y} w^y [x]_q^n d\mu_{-q}(y) = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} (-1)^l \frac{1}{1+wq^l}. \tag{6}$$

In special, we have

$$\varepsilon_{0,q}^w = \frac{[2]_q}{1+w}. \tag{7}$$

The idea for generalizing the fermionic integral is replacing the fermionic Haar measure with weakly (strongly) fermionic measure $\mathbb{Z}_p$ satisfying

$$\left|\mu_{-1}(a + p^n\mathbb{Z}_p) - \mu_{-1}(a + p^{n+1}\mathbb{Z}_p)\right|_p \leq \delta_{n,q}, \tag{8}$$

where $\delta_{n,q} \to 0$, a is a element of $\mathbb{Z}_p$, and $\delta_{n,q}$ is independent of a (for strongly fermionic measure, $\delta_{n,q}$ is replaced by Cp^{-n}, where C is a positive constant)(see [5, 7, 8, 13]).

Let $f(x)$ be a function defined on $\mathbb{Z}_p$. The fermionic integral of f with respect to a weakly fermionic measure μ_{-1} is

$$\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{n\to\infty} \sum_{x=0}^{p^n-1} f(x)\mu_{-1}(x+p^n\mathbb{Z}_p),$$

if the limit exists.

If μ_{-1} is a weakly fermionic measure on $\mathbb{Z}_p$, then the Radon–Nikodym derivative of μ_{-1} with respect to the Haar measure on $\mathbb{Z}_p$ as follows:

$$f_{\mu_{-1}}(x) = \lim_{n\to\infty} \mu_{-1}(x+p^n\mathbb{Z}_p), \tag{9}$$

(see [5, 7]).

Note that $f_{\mu_{-1}}$ is only a continuous function on $\mathbb{Z}_p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, let us define $\mu_{-1,f}$ as follows:

$$\mu_{-1,f}(x+p^n\mathbb{Z}_p) = \int_{x+p^n\mathbb{Z}_p} f(x)d\mu_{-1}(x), \tag{10}$$

where the integral is the fermionic p-adic invariant integral. From (9), we can easily note that $\mu_{-1,f}$ is a strongly fermionic measure on $\mathbb{Z}_p$(see [5, 7]). Since

$$\begin{aligned} \left|\mu_{-1,f}(x+p^n\mathbb{Z}_p) - \mu_{-1,f}(x+p^{n+1}\mathbb{Z}_p)\right|_p &= \Big| \sum_{x=0}^{p^n-1} f(x)(-1)^x - \sum_{x=0}^{p^n} f(x)(-1)^x \Big|_p \\ &= \left|\frac{f(p^n)}{p^n}\right| |p^n| \leq Cp^{-n}, \end{aligned}$$

where C is positive consatnt.

In this paper, we will give the Lebesgue–Radon–Nikodym theorem with respect to weighted and twisted p-adic q-measure on $\mathbb{Z}_p$. In special case, if there is no twisted, then we can derive the same result as Jeong and Rim, 2012 (see [5]). In the case of weight zero and no twisted, then we derive the same result as Kim, 2012 (see [7]).

2 The Lebesgue–Radon–Nikodym theorem with respect to the weighted p-adic q-measure

For any positive integer a and n with $a < p^n$ and $f \in UD(\mathbb{Z}_p)$, we define $\tilde{\mu}^w_{f,-q}$, weighted and twisted ferminonic measure on $\mathbb{Z}_p$ as follows:

$$\tilde{\mu}^w_{f,-q}(a+p^n\mathbb{Z}_p) = \int_{a+p^n\mathbb{Z}_p} w^x q^x f(x)d\mu_{-1}(x) \tag{11}$$

where the integral is the fermionic p-adic invariant integral on $\mathbb{Z}_p$.

From (11), we note that

$$\begin{aligned}\tilde{\mu}^w_{f,-q}(a+p^n\mathbb{Z}_p) &= \lim_{m\to\infty}\sum_{x=0}^{p^m-1} f(a+p^nx)(-1)^{a+p^nx}q^{a+p^nx}w^{a+p^nx} \\ &= (-1)^a w^a q^a \lim_{m\to\infty}\sum_{x=0}^{p^{m-n}-1} f(a+p^nx)(-1)^x q^{p^nx} \\ &= (-1)^a \int_{\mathbb{Z}_p} w^a f(a+p^nx)q^{a+p^nx}d\mu_{-1}(x).\end{aligned} \tag{12}$$

By (12), we get

$$\tilde{\mu}^w_{f,-q}(a+p^n\mathbb{Z}_p) = (-1)^a \int_{\mathbb{Z}_p} w^a f(a+p^nx)q^{a+p^nx}d\mu_{-1}(x). \tag{13}$$

Thus, by (13), we have

$$\tilde{\mu}_{\alpha f+\beta g,-q} = \alpha\tilde{\mu}_{f,-q} + \beta\tilde{\mu}_{g,-q}, \tag{14}$$

where $f, g \in UD(\mathbb{Z}_p)$ and α, β are positive constants.

By (11), (12), (13) and (14), we get

$$\left|\tilde{\mu}^w_{f,-q}(a+p^n\mathbb{Z}_p)\right|_p \leq \|f_{wq}\|_\infty, \tag{15}$$

where $\|f_{wq}\|_\infty = \sup_{x\in\mathbb{Z}_p}|f(x)q^xw^x|_p$.

Let $P(x) \in \mathbb{C}_p[[x]_q]$ be an arbitrary q-polynomial. Now we show $\tilde{\mu}^w_{P,-q}$ is a strongly weighted and twisted fermionic p-adic invariant measure on $\mathbb{Z}_p$. Without a loss of generality, it is enough to prove the statement for $P(x) = [x]^k_q$.

For $a \in \mathbb{Z}$ with $0 \leq a < p^n$, we have

$$\tilde{\mu}^w_{P,-q}(a+p^n\mathbb{Z}_p) = \lim_{m\to\infty}(-q)^a w^a \sum_{i=0}^{p^{m-n}-1} q^{p^ni}[a+ip^n]^k_q(-1)^i. \tag{16}$$

Note that

$$[a+ip^n]^k_q = ([a]_q + q^a[p^n]_q[i]_{q^{p^n}})^k. \tag{17}$$

By (6), (16) and (17),

$$\tilde{\mu}^w_{p,-q} = \left\{[a]^k_q\tilde{\varepsilon}_{0,q^{p^n}} + k[a]^{k-1}_q q^a[p^n]_q[i]_{q^{p^n}} + \cdots + q^{ak}[p^n]^k_q[i]^k_{q^{p^n}}\right\}.$$

By (7), (12) and (13), we easily get

$$\begin{aligned}\tilde{\mu}^w_{P,-q}(a+p^n\mathbb{Z}_p) &\equiv (-1)^a q^a w^a [a]^k_q \varepsilon^{w^{p^n}}_{0,q^{p^n}} \pmod{[p^n]_q} \\ &\equiv (-1)^a \frac{1+q^{p^n}}{1+w^{p^n}} P(a)q^aw^a \pmod{[p^n]_q}.\end{aligned} \tag{18}$$

For $x \in \mathbb{Z}_p$, let $x \equiv x_n \pmod{p^n}$ and $x \equiv x_{n+1} \pmod{p^{n+1}}$, where x_n, $x_{n+1} \in \mathbb{Z}$ with $0 \leq x_n < p^n$ and $0 \leq x_{n+1} < p^{n+1}$. Then we have

$$\left|\tilde{\mu}^w_{P,-q}(a+p^n\mathbb{Z}_p) - \tilde{\mu}^w_{P,-q}(a+p^{n+1}\mathbb{Z}_p)\right|_p \leq Cp^{-v_p(1-q^{p^n})}, \tag{19}$$

where C is positive constant and $n \gg 0$.

Let

$$f_{\tilde{\mu}^w_{P,-q}}(a) = \lim_{n\to\infty} \tilde{\mu}^w_{P,-q}(a + p^n\mathbb{Z}_p) = (-1)^a q^a w^a.$$

Then, by (18) and (19), we see that

$$f_{\tilde{\mu}^w_{P,-q}}(a) = (-1)^a q^a w^a [a]_q^k = (-1)^a q^a w^a P(a). \tag{20}$$

Since $f_{\tilde{\mu}^w_{P,-q}}(x)$ is continuous function on $\mathbb{Z}_p$, for $x \in \mathbb{Z}_p$, we have

$$f_{\tilde{\mu}^w_{P,-q}}(x) = (-1)^x q^x w^x [x]_q^k, (k \in \mathbb{Z}_+). \tag{21}$$

Let $g \in UD(\mathbb{Z}_p)$. Then, by (19), (20) and (21), we get

$$\begin{aligned}
\int_{\mathbb{Z}_p} g(x) d\tilde{\mu}^w_{P,-q}(x) &= \lim_{n\to\infty} \sum_{x=0}^{p^n-1} g(x) \tilde{\mu}^w_{P,-q}(x + p^n\mathbb{Z}_p) \\
&= \lim_{n\to\infty} \sum_{x=0}^{p^n-1} g(x) q^x w^x [x]_q^k (-1)^x \\
&= \int_{\mathbb{Z}_p} g(x) q^x w^x [x]_q^k d\mu_{-1}(x).
\end{aligned} \tag{22}$$

Therefore, by (22), we obtain the following theorem.

Theorem 2.1. *Let $P(x) \in \mathbb{C}_p[[x]_q]$ be an arbitrary q-polynomial. Then $\tilde{\mu}^w_{P,-q}$ is a strongly weighted and twisted fermionic p-adic invariant measure on $\mathbb{Z}_p$. Then we have*

$$f_{\tilde{\mu}^w_{P,-q}}(x) = (-1)^x q^x w^x P(x) \quad \textit{for all} \quad x \in \mathbb{Z}_p.$$

Furthermore, for any $g \in UD(\mathbb{Z}_p)$,

$$\int_{\mathbb{Z}_p} g(x) d\tilde{\mu}^w_{P,-q}(x) = \int_{\mathbb{Z}_p} g(x) P(x) q^x w^x d\mu_{-1}(x),$$

where the second integral is weighted and twisted fermionic p-adic invariant integral on $\mathbb{Z}_p$.

Let $f(x) = \sum_{n=0}^{\infty} a_{n,q} \binom{x}{n}_q$ be the Mahler q-expansion of continuous function on $\mathbb{Z}_p$, where

$$\binom{x}{n}_q = \frac{[x]_q [x-1]_q \cdots [x-n+1]_q}{[n]_q!}.$$

Then we note that $\lim_{n\to\infty} |a_{n,q}| = 0$.

Let

$$f_m(x) = \sum_{i=0}^{m} a_{i,q} \binom{x}{i}_q \in \mathbb{C}_p[[x]_q].$$

Then

$$\|(f - f_m)_{qw}\|_\infty \leq \sup_{n \leq m} |a_{n,q}|. \tag{23}$$

The function $f(x)$ can be rewritten as $f = f_m + f - f_m$. Thus, by (14) and (23), we get

$$\begin{aligned}&\big|\tilde{\mu}^w_{f,-q}(a+p^n\mathbb{Z}_p) - \tilde{\mu}^w_{f,-q}(a+p^{n+1}\mathbb{Z}_p)\big| \\ &\leq \max\{\big|\tilde{\mu}^w_{f_m,-q}(a+p^n\mathbb{Z}_p) - \tilde{\mu}^w_{f_m,-q}(a+p^{n+1}\mathbb{Z}_p)\big|, \\ &\big|\tilde{\mu}^w_{f-f_m,-q}(a+p^n\mathbb{Z}_p) - \tilde{\mu}^w_{f-f_m,-q}(a+p^{n+1}\mathbb{Z}_p)\big|\}\end{aligned} \tag{24}$$

From Theorem 2.1., we note that

$$\big|\tilde{\mu}^w_{f-f_m,-q}(a+p^n\mathbb{Z}_p)\big|_p \leq \|f-f_m\|_\infty \leq C_1 p^{-2v_p(1-q^{p^n})}, \tag{25}$$

where C_1 are positive constants. For $m \gg 0$, we have $\|f\|_\infty = \|f_m\|_\infty$. So, we see that

$$\begin{aligned}&\big|\tilde{\mu}^w_{f_m,-q}(a+p^n\mathbb{Z}_p) - \tilde{\mu}^w_{f_m,-q}(a+p^{n+1}\mathbb{Z}_p)\big|_p \\ &= \big|\frac{f_m([p^n]_q)q^{p^n}}{[p^n]_q^2}\big|_p\big|[p^n]_q^2\big|_p \leq \|f_m q^x w^x\|_\infty \big|[p^n]_q^2\big|_p \leq C_2 p^{-2v_p(1-q^{p^n})},\end{aligned} \tag{26}$$

where C_2 is a positive constant.

By (25), we get

$$\begin{aligned}&\big|(-1)^a q^a w^a f(a) - \tilde{\mu}^w_{f,-q}(a+p^n\mathbb{Z}_p)\big|_p \\ &\leq \max\{|q^a||w_a|\big|q^a w^a f(a) - f_m(a)q^a w^a\big|_p, \big|q^a w^a f_m(a) - \tilde{\mu}^w_{f_m,-q}(a+p^n\mathbb{Z}_p)\big|_p, \\ &\big|\tilde{\mu}^w_{f-f_m,-q}(a+p^n\mathbb{Z}_p)\big|_p\} \\ &\leq \max\{|q^a||w_a|\big|f(a) - f_m(a)\big|_p, \big|f_m(a) - \tilde{\mu}^w_{f_m,-q}(a+p^n\mathbb{Z}_p)\big|_p, \|(f-f_m)_{qw}\|_\infty\}\end{aligned} \tag{27}$$

Let us assume that fix $\epsilon > 0$, and fix m such that $\|f - f_m\| < \epsilon$. Then we have

$$\big|(-q)^a w^a f(a) - \tilde{\mu}^w_{f,-q}(a+p^n\mathbb{Z}_p)\big|_p \leq \epsilon \quad \text{for} \quad n \gg 0. \tag{28}$$

Thus, by (28), we have

$$f_{\tilde{\mu}^w_{f,-q}}(a) = \lim_{n\to\infty} \tilde{\mu}^w_{f,-q}(a+p^n\mathbb{Z}_p) = (-1)^a q^a w^a f(a). \tag{29}$$

Let m be the sufficiently large number such that $\|f - f_m\|_\infty \leq p^{-n}$. Then we get

$$\begin{aligned}\tilde{\mu}^w_{f,-q}(a+p^n\mathbb{Z}_p) &= \tilde{\mu}^w_{f_m,-q}(a+p^n\mathbb{Z}_p) + \tilde{\mu}^w_{f-f_m,-q}(a+p^n\mathbb{Z}_p) \\ &= (-1)^a q^a w^a f(a) \pmod{[p^n]_q^2}.\end{aligned}$$

For $g \in UD(\mathbb{Z}_p)$, we have

$$\int_{\mathbb{Z}_p} g(x)d\tilde{\mu}^w_{f,-q}(x) = \int_{\mathbb{Z}_p} f(x)g(x)\frac{[2]_q}{1+w}q^x w^x d\mu_{-1}(x).$$

Let f be the function from $UD(\mathbb{Z}_p)$ to $Lip(\mathbb{Z}_p)$. We easily see that $w^x q^x \mu_{-1}(x+p^n\mathbb{Z}_p)$ is a strongly weighted and twisted p-adic invariant measure on $\mathbb{Z}_p$ and

$$\big|(f_{qw})_{\mu_{-1}}(a) - w^a q^a \mu_{-1}(a+p^n\mathbb{Z}_p)\big|_p \leq C_3 p^{-2v_p(1-q^{p^n})},$$

where $f_{qw}(x) = q^x w^x f(x)$ and C_3 is positive constant and $n \in \mathbb{Z}_+$.

If $\mu_{1,-q}^w$ is associated with strongly weighted and twisted fermionic invarinat measure on $\mathbb{Z}_p$, then we have

$$\left|\tilde{\mu}_{1,-q}^w(a + p^n\mathbb{Z}_p) - (f_{qw})_{\mu_{-1}}(a)\right|_p \le C_4 p^{-2v_p(1-q^{p^n})},$$

where $n > 0$ and C_4 is positive constant.

For $n \gg 0$, we have

$$\begin{aligned} &\left|q^a w^a \mu_{-1}(a + p^n\mathbb{Z}_p) - \tilde{\mu}_{1,-q}^w(a + p^n\mathbb{Z}_p)\right|_p \\ &\le \left|q^a w^a \mu_{-1}(a + p^n\mathbb{Z}_p) - (f_{qw})_{\mu_{-1}^w}(a)\right|_p + \left|(f_{qw})_{\mu_{-1}}(a) - \tilde{\mu}_{1,-q}^w(a + p^n\mathbb{Z}_p)\right|_p \le K, \end{aligned} \tag{30}$$

where K is positive constant.

Hence, $wq\mu_{-1} - \tilde{\mu}_{1,-q}^w$ is a weighted and twisted measure on $\mathbb{Z}_p$. Therefore, we obtain the following theorem.

Theorem 2.2. *Let $wq\mu_{-1}$ be a strongly weighted and twisted p-adic invariant measure on $\mathbb{Z}_p$, and assume that the fermionic weighted and twisted Radon–Nikodym derivative $(f_{qw})_{\mu_{-1}}$ on $\mathbb{Z}_p$ is uniformly differentiable function. Suppose that $\tilde{\mu}_{1,-q}^w$ is the strongly weighted and twisted fermionic p-adic invariant measure associated with $(f_{qw})_{\mu_{-1}}$. Then there exists a weighted and twisted measure $\tilde{\mu}_{2,-q}^w$ on $\mathbb{Z}_p$ such that*

$$w^x q^x \mu_{-1}(x + p^n\mathbb{Z}_p) = \tilde{\mu}_{1,-q}^w(x + p^n\mathbb{Z}_p) + \tilde{\mu}_{2,-q}^w(x + p^n\mathbb{Z}_p).$$

3 Conclusion

The Theorem 2.2. is the version of the Lebesgue–Radon–Nikodym theorem with respect to weighted and twisted p-adic q-measure on $\mathbb{Z}_p$. In special case, if there is no twisted, then we can derive the same result as Jeong and Rim, 2012 (see [5]). In the case of weight zero and no twisted, then we derive the same result as Kim, 2012 (see [7]).

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