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Identities involving q-Genocchi numbers and polynomials

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Abstract: In this paper, we focus on the q-Genocchi numbers and polynomials. We introduce new identities of the q-Genocchi numbers and polynomials by using the fermionic p-adic integral on $\mathbb{Z}_p$. Also, we give Cauchy-integral formula for the q-Genocchi polynomials and derive the distribution formula q-Genocchi polynomials by using measure theory on p-adic integral. Finally, we get q-Zeta-type function by using Mellin transformation (sometimes known as Laplace transformation) and show that this function interpolates to the q-Genocchi polynomials at *negative integers*.

Keywords: Genocchi numbers and polynomials, q-Genocchi numbers and polynomials, p-adic q-integral on $\mathbb{Z}_p$, Mellin transformation, q-Zeta function.
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1 Preliminaries

Recently, Kim and Lee have given some properties for the q-Euler numbers and polynomials in [7]. This type numbers and polynomials and their various generalizations have been studied in several different ways for a long time (see [1–‘21] for a systematic work).

By using p-adic q-integral on $\mathbb{Z}_p$, Kim defined many new generating functions of the q-Bernoulli poynomials, q-Euler polynomials and q-Genocchi polynomials in his arithmetic works (for details, see [6–19]). The works of have been benefited for further works of many mathematicians in deriving new interesting identities of some special polynomials. Actually, we are motivated from his inspiring works to write this paper.

Assume that p is a fixed odd prime number. Throughout this work, we make use of some notations as follows: $\mathbb{Q}_p$ denotes the field p-adic rational numbers and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. That is,

$$\mathbb{Q}_p = \left\{ x : x = \sum_{n=-k}^{\infty} a_n p^n \text{ with } 0 \leq a_n \leq p-1 \right\}.$$

Then $\mathbb{Z}_p$ is integral domain which is defined by

$$\mathbb{Z}_p = \left\{ x : x = \sum_{n=0}^{\infty} a_n p^n \text{ with } 0 \leq a_n \leq p-1 \right\}$$

or

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \leq 1 \right\}.$$

We assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ as an indeterminate. The p-adic absolute value $|.|_p$, is normally given by

$$|x|_p = p^{-r}$$

where $x = p^r \frac{s}{t}$ with $(p,s) = (p,t) = (s,t) = 1$ and $r \in \mathbb{Q}$.

The q-extension of x with the display notation of $[x]_q$ is introduced by

$$[x]_q = \frac{1-q^x}{1-q}.$$

We want to note that $\lim_{q\to 1} [x]_q = x$ (see[1-22]). Also, we make use of the notation $\mathbb{N}^*$ that stands of combining with zero and natural numbers.

We consider that η is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the difference quotient

$$\Phi_\eta (x,y) = \frac{\eta(x) - \eta(y)}{x-y},$$

have a limit $\eta'(a)$ as $(x,y) \to (a,a)$ and denote this by $\eta \in UD(\mathbb{Z}_p)$. Then, for $\eta \in UD(\mathbb{Z}_p)$, we can discuss the following

$$\frac{1}{[p^n]_q} \sum_{0 \leq \xi < p^n} \eta(\xi) q^\xi = \sum_{0 \leq \xi < p^n} \eta(\xi) \mu_q (\xi + p^n \mathbb{Z}_p),$$

which represents as a p-adic q-analogue of Riemann sums for η. The integral of η on $\mathbb{Z}_p$ will be defined as the limit $(n \to \infty)$ of these sums, when it exists. The p-adic q-integral of function $\eta \in UD(\mathbb{Z}_p)$ is defined by T. Kim in [6, 10, 15] by

$$I_q(\eta) = \int_{\mathbb{Z}_p} \eta(\xi) \, d\mu_q(\xi) = \lim_{n\to\infty} \frac{1}{[p^n]_q} \sum_{\xi=0}^{p^n-1} \eta(\xi) \, q^{\xi}. \tag{1.1}$$

The bosonic integral is considered as a bosonic limit $q \to 1$, $I_1(\eta) = \lim_{q\to 1} I_q(\eta)$. Similarly, the fermionic p-adic integral on $\mathbb{Z}_p$ is introduced by T. Kim as follows:

$$I_{-q}(\eta) = \int_{\mathbb{Z}_p} \eta(\xi) \, d\mu_{-q}(\xi) \tag{1.2}$$

(for more details, see [16–18]). Obviously that

$$\lim_{q\to 1} I_{-q}(\eta) = I_{-1}(\eta) = \int_{\mathbb{Z}_p} \eta(\xi) \, d\mu_{-1}(\xi) = \lim_{n\to\infty} \sum_{\xi=0}^{p^n-1} \eta(\xi) (-1)^{\xi}. \tag{1.3}$$

From (1.3), it is well-known as the useful property for the fermionic p-adic q-integral on $\mathbb{Z}_p$:

$$I_{-1}(\eta_1) + I_{-1}(\eta) = 2\eta(0), \tag{1.4}$$

where $\eta_1(x) = \eta(x+1)$ (for details, see[2-4, 11-14, 16-22]).

The q-Genocchi polynomials are considered by

$$G_{n,q}(x) = n \int_{\mathbb{Z}_p} q^{\xi} (x+\xi)^{n-1} \, d\mu_{-1}(\xi). \tag{1.5}$$

From (1.5), we have

$$G_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^l G_{n-l,q}$$

where $G_{n,q}(0) := G_{n,q}$ are called q-Genocchi numbers. Then, q-Genocchi numbers can be given by

$$G_{0,q} = 0 \text{ and } q(G_q + 1)^n + G_{n,q} = \begin{cases} 2, & \text{if } n = 1 \\ 0, & \text{if } n \geq 1 \end{cases}$$

with the usual convention about replacing $(G_q)^n$ by $G_{n,q}$ is used (for details, see [4]).

Our objective in the present paper is to derive not only new but also novel and interesting properties of the q-Genocchi numbers and polynomials. Our applications for the q-Genocchi polynomials seem to be useful in mathematics for engineerings (on this subject, see [23]).

2 Some properties on the q-Genocchi numbers and polynomials

Let $\eta(x) = q^{\xi} e^{t(x+\xi)}$. Then, by using (1.4), we see that

$$t \int_{\mathbb{Z}_p} q^{\xi} e^{t(x+\xi)} d\mu_{-1}(\xi) = \frac{2t}{qe^t + 1} e^{xt}.$$

From the last equality and (1.5), we easily derive the following generating function of the q-Genocchi polynomials:

$$t\int_{\mathbb{Z}_p} q^{\xi}e^{t(x+\xi)}d\mu_{-1}(\xi)=\sum_{n=0}^{\infty}G_{n,q}(x)\frac{t^n}{n!}=\frac{2t}{qe^t+1}e^{xt},\ |\log q+t|<\pi. \tag{2.1}$$

Substituting $x\to x+y$ into (2.1), then we write

$$\begin{aligned}\sum_{n=0}^{\infty}G_{n,q}(x+y)\frac{t^n}{n!}&=\frac{2t}{qe^t+1}e^{(x+y)t}\\&=\left(\sum_{n=0}^{\infty}G_{n,q}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}y^n\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}G_{k,q}(x)\,y^{n-k}\right)\frac{t^n}{n!}\end{aligned}$$

From the above, we easily express the following theorem:

Theorem 1. *The following holds:*

$$G_{n,q}(x+y)=\sum_{k=0}^{n}\binom{n}{k}G_{k,q}(x)\,y^{n-k}. \tag{2.2}$$

By (2.2), we consider the following

$$G_{n,q}(x+y)=\frac{2}{[2]_q}ny^{n-1}+\sum_{k=2}^{n}\binom{n}{k}G_{k,q}(x)\,y^{n-k}.$$

From this, it follows that

$$G_{n,q}(x+y)-\frac{2}{[2]_q}ny^{n-1}=\sum_{k=2}^{n}\binom{n}{k}G_{k,q}(x)\,y^{n-k}$$

can be derived and so we reach the following theorem:

Theorem 2. *For $n\in\mathbb{N}^*$, one has*

$$\begin{aligned}&\sum_{k=0}^{n}\frac{\binom{n}{k}}{(k+2)(k+1)}G_{k+2,q}(x)\,y^{n-k}\\&=\frac{G_{n+2,q}(x+y)-\frac{2}{[2]_q}(n+2)\,y^{n+1}}{(n+2)(n+1)}.\end{aligned} \tag{2.3}$$

Replacing y by $-y$ into (2.3), then we get

$$\begin{aligned}&\frac{G_{n+2,q}(x-y)-\frac{2}{[2]_q}(n+2)(-1)^{n+1}y^{n+1}}{(n+2)(n+1)}\\&=\sum_{k=0}^{n}\frac{\binom{n}{k}(-1)^{n-k}}{(k+2)(k+1)}G_{k+2,q}(x)\,y^{n-k}.\end{aligned} \tag{2.4}$$

By (2.4), it follows that

$$\sum_{k=0}^{n}\frac{\binom{n}{k}\left(-1\right)^{k}}{\left(k+2\right)\left(k+1\right)}G_{k+2,q}\left(x\right)y^{n-k} = \frac{\left(-1\right)^{n}G_{n+2,q}\left(x-y\right)+\frac{2}{[2]_q}\left(n+2\right)y^{n+1}}{\left(n+1\right)\left(n+2\right)}. \tag{2.5}$$

Therefore, from the expressions of (2.3) and (2.5), we procure the following theorem:

Theorem 3. *The following holds true:*

$$\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\binom{n}{2k}}{\left(k+1\right)\left(2k+1\right)}G_{2k+2,q}\left(x\right)y^{n-2k} = \frac{\left(-1\right)^{n}G_{n+2,q}\left(x-y\right)+G_{n+2,q}\left(x+y\right)}{\left(n+1\right)\left(n+2\right)} \tag{2.6}$$

where $[.]$ *is Gauss' symbol.*

By (2.4), we have the following identity

$$\sum_{k=2}^{n}\frac{\binom{n}{k}\left(-1\right)^{k}}{k\left(k-1\right)}G_{k,q}\left(x\right)y^{n-k} = \frac{\left(-1\right)^{n}G_{n,q}\left(x-y\right)+\frac{2}{[2]_q}ny^{n-1}}{n\left(n-1\right)}. \tag{2.7}$$

By (2.4), (2.5) and (2.7), then we have the following theorem:

Theorem 4. *For* $n\in\mathbb{N}^{*}$*, we get*

$$\frac{\left(-1\right)^{n}G_{n+2,q}\left(x-y\right)+G_{n+2,q}\left(x+y\right)}{\left(n+2\right)\left(n+1\right)} = \sum_{k=0}^{\left[\frac{n+1}{2}\right]}\frac{\binom{n}{2k}}{\left(k+1\right)\left(2k+1\right)}G_{2k+2,q}\left(x\right)y^{n-2k}. \tag{2.8}$$

Taking $y=1$ into (2.3), it follows that

$$q\sum_{k=0}^{n}\frac{\binom{n}{k}}{\left(k+2\right)\left(k+1\right)}G_{k+2,q}\left(x\right)=\frac{qG_{n+2,q}\left(x+1\right)}{\left(n+1\right)\left(n+2\right)}-\frac{2q}{[2]_q\left(n+1\right)}. \tag{2.9}$$

We need the following for sequel of this paper:

$$2e^{tx}=\frac{1}{t}\left(q\frac{2t}{qe^{t}+1}e^{(x+1)t}+\frac{2t}{qe^{t}+1}e^{xt}\right).$$

From the above, we easily develop the following:

$$qG_{n+1,q}\left(x+1\right)+G_{n+1,q}\left(x\right)=\left(n+1\right)2x^{n}. \tag{2.10}$$

By (2.9) and (2.10), we state the following theorem:

Theorem 5. *The following holds:*

$$\sum_{k=0}^{n}\frac{\binom{n}{k}}{(k+2)(k+1)}G_{k+2,q}(x) \tag{2.11}$$
$$=\frac{2x^{n+1}}{qn+q}-\frac{G_{n+2,q}(x)}{(qn+q)(n+2)}-\frac{2}{[2]_q(n+1)}.$$

Thanks to equality of $\lim_{q\to 1}G_{n,q}(x)=G_{n,1}(x):=G_n(x)$, where $G_n(x)$ are known as Genocchi polynomials which is given via the exponential generating function, as follows:

$$\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}=\frac{2t}{e^t+1}e^{xt}\ \ (|t|<\pi),$$

(see [1–4, 12, 13, 21]). From the above, as $q\to 1$ in (2.11), we discover the following corollary:

Corollary 1. *The following*

$$\sum_{k=0}^{n}\frac{\binom{n}{k}}{(k+2)(k+1)}G_{k+2}(x)$$
$$=\frac{2x^{n+1}}{n+1}-\frac{G_{n+2}(x)}{(n+1)(n+2)}-\frac{1}{n+1}$$

is true.

Let us take $y=1$ and $n\to 2n$ into (2.6), becomes

$$\sum_{k=0}^{n}\frac{\binom{2n}{2k}}{(k+1)(2k+1)}G_{2k+2,q}(x) \tag{2.12}$$
$$=\frac{G_{2n+2,q}(x-1)+G_{2n+2,q}(x+1)}{(2n+1)(2n+2)}$$
$$=\frac{\frac{1}{q}(qG_{2n+2,q}(x+1)+G_{2n+2,q}(x))+qG_{2n+2,q}(x)+G_{2n+2,q}(x-1)}{(2n+1)(2n+2)}$$
$$-\frac{G_{2n+2,q}(x)}{q(2n+1)(2n+2)}-\frac{qG_{2n+2,q}(x)}{(2n+1)(2n+2)}$$
$$=\frac{2(n+2)x^{n+1}}{(2n+1)(2n+2)}+\frac{2(n+2)(x-1)^{n+1}}{(2n+1)(2n+2)}-\frac{G_{2n+2,q}(x)}{q(2n+1)(2n+2)}-\frac{qG_{2n+2,q}(x)}{(2n+1)(2n+2)}$$

After these applications, we conclude with the following theorem:

Theorem 6. *The following identity*

$$\sum_{k=0}^{n}\frac{\binom{2n}{2k}}{(k+1)(2k+1)}G_{2k+2,q}(x) \tag{2.13}$$
$$=\frac{(n+2)x^{n+1}}{(2n+1)(n+1)}+\frac{(n+2)(x-1)^{n+1}}{(2n+1)(n+1)}-\frac{G_{2n+2,q}(x)}{q(2n+1)(2n+2)}-\frac{qG_{2n+2,q}(x)}{(2n+1)(2n+2)}$$

is true.

Now, we analyse as $q \to 1$ for the equation (2.13) and so we readily state the following corollary which seems to be interesting property for the Genocchi polynomials.

Corollary 2. *The following equality holds:*

$$\sum_{k=0}^{n} \frac{\binom{2n}{2k}}{(k+1)(2k+1)} G_{2k+2}(x) \tag{2.14}$$

$$= \frac{2(n+2)x^{n+1}}{(2n+1)(2n+2)} + \frac{2(n+2)(x-1)^{n+1}}{(2n+1)(2n+2)} - \frac{G_{2n+2}(x)}{(2n+1)(2n+2)} - \frac{G_{2n+2}(x)}{(2n+1)(2n+2)}.$$

Substituting $n \to 2n+1$ and $y=1$ into (2.8), we compute

$$\sum_{k=0}^{n} \frac{\binom{2n+1}{2k}}{(k+1)(2k+1)} G_{2k+2,q}(x)$$

$$= \frac{G_{2n+3,q}(x+1) - G_{2n+3,q}(x-1)}{(2n+3)(2n+2)}$$

$$= \frac{\frac{1}{q}\left(qG_{2n+3,q}(x+1) + G_{2n+3,q}(x)\right) - \left(G_{2n+3,q}(x) + G_{2n+3,q}(x-1)\right)}{(2n+3)(2n+2)}$$
$$+ \left(\frac{q-1}{q}\right) \frac{G_{2n+3,q}(x)}{(2n+3)(2n+2)}$$

$$= \frac{x^{2n+2}}{q(n+1)} - \frac{(x-1)^{2n+2}}{(n+1)} + \left(\frac{q-1}{q}\right) \frac{G_{2n+3,q}(x)}{(2n+3)(2n+2)}.$$

Therefore, we obtain the following theorem:

Theorem 7. *The following equality holds:*

$$\sum_{k=0}^{n} \frac{\binom{2n+1}{2k}}{(k+1)(2k+1)} G_{2k+2,q}(x)$$

$$= \frac{x^{2n+2}}{q(n+1)} - \frac{(x-1)^{2n+2}}{(n+1)} + \left(\frac{q-1}{q}\right) \frac{G_{2n+3,q}(x)}{(2n+3)(n+1)}.$$

As $q \to 1$ in the above theorem, then we easily derive the following corollary:

Corollary 3. *For $n \in \mathbb{N}^*$, then we have*

$$\sum_{k=0}^{n} \frac{\binom{2n+1}{2k}}{(k+1)(2k+1)} G_{2k+2,q}(x)$$

$$= \frac{2x^{2n+2}}{(2n+2)} - \frac{2(x-1)^{2n+2}}{(2n+2)}.$$

3 Conclusion

In this final section, we recall the generating function of the q-Genocchi polynomials, as follows:

$$\mathcal{F}_q(x,t) = \frac{2t}{qe^t+1}e^{xt} = \sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!}. \tag{3.1}$$

Using the definition of the k-th derivative as $\frac{d^k}{dt^k}$ to (3.1), then we easily see that

$$\frac{d^k}{dt^k}\left(\frac{2t}{qe^t+1}e^{xt}\right) = \frac{d^k}{dt^k}\left(\sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!}\right). \tag{3.2}$$

By applying $\lim_{t\to 0}$ on the both sides in (3.2), then we conclude with the following theorem:

Theorem 8. *The following identity*

$$G_{k,q}(x) = \lim_{t\to 0}\left[\frac{d^k}{dt^k}\left(\frac{2t}{qe^t+1}e^{xt}\right)\right] \tag{3.3}$$

is true.

We now consider Cauchy-integral formula of the q-Genocchi polynomials which is a vital and important in complex analysis, is an important statement about line integrals for holomorphic functions in the complex plane. So, by using the equation of (3.3), we can state the following theorem:

Theorem 9. *The following Cauchy-integral holds true:*

$$G_{n,q}(x) = \frac{n!}{2\pi i}\int_C \mathcal{F}_q(x,t)\frac{dt}{t^{n+1}}$$

where C is a loop which starts at $-\infty$, encircles the origin once in the positive direction, and the returns $-\infty$.

Distribution formula for the q-Genocchi polynomials is important to study regarding p-adic Measure theory. That is,

$$\begin{aligned}
\int_{\mathbb{Z}_p} q^y (x+y)^n \, d\mu_{-1}(y) &= \lim_{n\to\infty}\sum_{\xi=0}^{dp^n-1}(-1)^{\xi}(x+\xi)^n q^{\xi} \\
&= d^n\sum_{a=0}^{d-1}(-1)^a q^a\left(\lim_{n\to\infty}\sum_{\xi=0}^{p^n-1}(-1)^{\xi}\left(\frac{x+a}{d}+\xi\right)^n q^{d\xi}\right) \\
&= d^n\sum_{a=0}^{d-1}(-1)^a q^a\frac{G_{n+1,q}\left(\frac{x+a}{d}\right)}{n+1}.
\end{aligned}$$

After the above applications, we procure the following theorem.

Theorem 10. *For $n\in\mathbb{N}^*$, then we have*

$$G_{n,q}(dx) = d^{n-1}\sum_{a=0}^{d-1}(-1)^a q^a G_{n,q}\left(x+\frac{a}{d}\right).$$

By utilizing from the definition of the geometric series in (3.1), we easily see that

$$\sum_{m=0}^{\infty} G_{m,q}(x) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left(2(m+1) \sum_{n=0}^{\infty} (-1)^n q^n n^m \right) \frac{t^{m+1}}{(m+1)!},$$

by comparing the coefficients on the both sides, then we have, for $m \in \mathbb{N}$

$$\frac{G_{m+1,q}(x)}{m+1} = [2]_q \sum_{n=1}^{\infty} (-1)^n q^n n^m. \tag{3.4}$$

Now also, we develop the following applications by using Mellin transformation to the generating function of the q-Genocchi polynomaials: For $s \in \mathbb{C}$ and $\Re e(s) > 1$,

$$\begin{aligned} \zeta(s, x : q) &= \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-2} \left\{ -\mathcal{F}_q(x, -t) \right\} dt \qquad (3.5) \\ &= 2 \sum_{n=0}^{\infty} (-1)^n q^n \left(\int_0^{\infty} t^{s-1} e^{-nt} dt \right) \\ &= 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{n^s} \end{aligned}$$

Thus, we can state the definition of the q-Zeta-type function as follows:

$$\zeta(s, x : q) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{n^s}. \tag{3.6}$$

As $q \to 1$ in (3.6), turns into

$$\lim_{q \to 1} \zeta(s, x : q) = \zeta(s, x : 1) := \zeta(s, x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$$

which is well-known as Euler-Zeta function (see [6]). By (3.4) and (3.6), we get

$$\zeta(-n, x : q) = \frac{G_{n+1,q}(x)}{n+1}.$$

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