Notes on Number Theory and Discrete Mathematics
ISSN 1310–5132
Vol. 20, 2014, No. 5, 58–63

On some Pascal's like triangles. Part 7

Krassimir T. Atanassov

Department of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences,
Acad. G. Bonchev Str., Bl. 105, Sofia-1113, Bulgaria
e-mail: krat@bas.bg

Abstract: A series of Pascal's like triangles with different forms are described and some of their propertiesa are given.
Keywords: Pascal triangle, Sequence.
AMS Classification: 11B37.

In a series of the six papers [1–6], we discussed a new type of Pascal's like triangles. Triangles in the present form, but not with the present sense, are described in different publications, e.g. [7–10].

Now, we continue the research over the Pascal's like triangles from [6], where we discussed infinite triangles in the form

$$\begin{array}{ccccccc} & & & a_{1,1} & & & \\ & & a_{2,1} & a_{2,2} & a_{2,3} & & \\ & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} \\ & \vdots & & \vdots & & \vdots & \end{array}$$

where $a_{i,1}$ and $a_{i,2i-1}$ are arbitrary real (complex) numbers (i.e., without the condition to be equal, formulated in [1]) and for every natural number $i \geq 1$ and
1. for every natural number j for which $2 \leq j \leq i-1$ it will be valid:

$$a_{i,j} = a_{i,j-1} + a_{i-1,j-1};$$

2. for every natural number j for which $i+1 \leq j \leq 2i-1$ it will be valid:

$$a_{i,j} = a_{i,j+1} + a_{i-1,j-1};$$

3. for $i \geq 2$:

$$a_{i,i} = a_{i-1,i-1} + \frac{a_{i,i-1} + a_{i,i+1}}{2}.$$

Let each of the sequences $\{a_{i,1}\}_{i\geq 1}$ and $\{a_{i,2i-1}\}_{i\geq 1}$ be called a *generating sequence* and the sequences $\{a_{i,i}\}_{i\geq 1}$ be called a *generated sequence.*

Let us give two examples.

First, let the generating sequences be “1, 0, 1, 0, 1, ...” and “1, 2, 1, 2, 1, ...”. Then we obtain the triangle

$$\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & 0 & \mathbf{2} & 2 & & & & & \\
 & & & & 1 & 1 & \mathbf{4} & 3 & 1 & & & & \\
 & & & 0 & 1 & 2 & \mathbf{8} & 6 & 3 & 2 & & & \\
 & & 1 & 1 & 2 & 4 & \mathbf{16} & 12 & 6 & 3 & 1 & & \\
 & 0 & 1 & 2 & 4 & 8 & \mathbf{32} & 24 & 12 & 6 & 3 & 2 & \\
1 & 1 & 2 & 4 & 8 & 16 & \mathbf{64} & 48 & 24 & 12 & 6 & 3 & 1 \\
 & & & & \vdots & & \vdots & & \vdots & & & &
\end{array}$$

with generated sequence $\{2^n\}_{n\geq 0}$. It is interesting to mention that such sequence exists in triangle

$$\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & 1 & \mathbf{2} & 1 & & & & & \\
 & & & & 1 & 2 & \mathbf{4} & 2 & 1 & & & & \\
 & & & 1 & 2 & 4 & \mathbf{8} & 4 & 2 & 1 & & & \\
 & & 1 & 2 & 4 & 8 & \mathbf{16} & 8 & 4 & 2 & 1 & & \\
 & 1 & 2 & 4 & 8 & 16 & \mathbf{32} & 16 & 8 & 4 & 2 & 1 & \\
1 & 2 & 4 & 8 & 16 & 32 & \mathbf{64} & 32 & 16 & 8 & 4 & 2 & 1 \\
 & & & & \vdots & & \vdots & & \vdots & & & &
\end{array}$$

(see [1]).

Second, let the generating sequences be “1, 1, 1, ...” and $\{2^n - 1\}_{n\geq 1}$. Then we obtain the triangle

$$\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & 1 & \mathbf{3} & 3 & & & & & \\
 & & & & 1 & 2 & \mathbf{9} & 10 & 7 & & & & \\
 & & & 1 & 2 & 4 & \mathbf{27} & 32 & 22 & 15 & & & \\
 & & 1 & 2 & 4 & 8 & \mathbf{81} & 100 & 68 & 46 & 31 & & \\
 & 1 & 2 & 4 & 8 & 16 & \mathbf{243} & 308 & 208 & 140 & 94 & 63 & \\
1 & 2 & 4 & 8 & 16 & 32 & \mathbf{729} & 940 & 632 & 424 & 284 & 190 & 127 \\
 & & & & \vdots & & \vdots & & \vdots & & & &
\end{array}$$

with generated sequence $\{3^n\}_{n\geq 0}$. We again can mention that such sequence exists in triangle

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & \mathbf{1} & & & & & & & \\
 & & & & & & 2 & \mathbf{3} & 2 & & & & & & \\
 & & & & & 4 & 6 & \mathbf{9} & 6 & 4 & & & & & \\
 & & & & 8 & 12 & 18 & \mathbf{27} & 18 & 12 & 8 & & & & \\
 & & & 16 & 24 & 36 & 54 & \mathbf{81} & 54 & 36 & 24 & 16 & & & \\
 & & 32 & 48 & 72 & 108 & 162 & \mathbf{243} & 162 & 108 & 72 & 48 & 32 & & \\
 & 64 & 96 & 144 & 216 & 324 & 486 & \mathbf{729} & 486 & 324 & 216 & 144 & 96 & 64 & \\
128 & 192 & 288 & 432 & 648 & 972 & 1458 & \mathbf{2187} & 1458 & 972 & 648 & 432 & 288 & 192 & 128 \\
 & & & & & \vdots & & \vdots & & \vdots & & & & &
\end{array}
$$

(see [1]).

We can prove, e.g., by induction the following

Theorem: For every two generating sequences $\{a_{i,1}\}_{i\geq 1}$ and $\{a_{i,2i-1}\}_{i\geq 1}$, i.e., such that have a joint first element $a_{1,1}$, their generated sequence $\{a_{i,i}\}_{i\geq 1}$ has the form for every natural number $n \ (n \geq 2)$:

$$a_{n,n} = \frac{1}{2}\sum_{i=1}^{n} \binom{n-1}{i-1} (a_{i,1} + a_{i,2i-1}). \tag{1}$$

Proof: Following the above described procedure, we construct the triangle

$$
\begin{array}{ccccc}
 & & a_{1,1} & & \\
 & a_{2,1} & a_{1,1} + \frac{a_{2,1}+a_{2,3}}{2} & a_{2,3} & \\
a_{3,1} & a_{2,1} + a_{3,1} & a_{1,1} + a_{2,1} + a_{2,3} + \frac{a_{3,1}+a_{3,5}}{2} & a_{2,3} + a_{3,5} & a_{3,5} \\
 & \vdots & \vdots & \vdots &
\end{array}
$$

For $n = 2$, we see that (2) is valid. Let us assume that the assertion is valid for some natural number $n \geq 2$. We must check that it is valid for $n+1$, but before this, we must discuss the form of the elements of the n-th triangle row. From the above constructed triangle, we can see that the m-th member of this row ($1 \leq m \leq n-1$; i.e., the member is in the left side of the n-th row, where n is some natural number and without restriction, it can be the above fixed number) has the form

$$a_{n,m} = \sum_{i=1}^{m} \binom{m-1}{i-1} a_{n-m+i,1}. \tag{2}$$

Really, for $m = 1$, this assertion is valid. Let it be valid for some natural number $1 \leq m \leq n-2$.

Obviously, the elements of the triangle, that lie over each line parallel to the generating line (in the present case – the left generating line) are obtained in the equal way, e.g., members $a_{n,m}$ and $a_{n-1,m}$. Then by the induction assumption,

$$a_{n,m+1} = a_{n-1,m} + a_{n,m}$$

$$= \sum_{i=1}^{m} \binom{m-1}{i-1} a_{n-1-m+i,1} + \sum_{i=1}^{m-1} \binom{m-1}{i-1} a_{n-m+i,1}$$

$$= \binom{m-1}{0} a_{n-m,1} + \sum_{i=2}^{m} \binom{m-1}{i-1} a_{n-1-m+i,1}$$

$$+ \sum_{i=1}^{m-1} \binom{m-1}{i-1} a_{n-m+i,1} + \binom{m-1}{m-1} a_{n,1}$$

$$= \binom{m}{0} a_{n-m,1} + \sum_{i=2}^{m} \binom{m}{i} a_{n-(m+1)+i,1} + \binom{m}{m} a_{n,1}$$

$$= \sum_{i=1}^{m+1} \binom{m}{i-1} a_{n-(m+1)+i,1}.$$

Therefore, (2) is valid.

Now, we return to the basic proof.

The value of $a_{n,n}$ is given by induction assumption from (1). From (2), we obtain for the values of the n-th member of $(n+1)$-st row:

$$a_{n+1,n} = \sum_{i=1}^{n} \binom{n-1}{i-1} a_{i+1,1}. \tag{3}$$

Similarly, we see that the $(n+2)$-nd member of $(n+1)$-st row (i.e., the first member in the right side of the $(n+1)$-st row) has the form

$$a_{n+1,n+2} = \sum_{i=1}^{n} \binom{n-1}{i-1} a_{i+1,2i+1}. \tag{4}$$

Therefore, we can calculate the value of member $a_{n+1,n+1}$, using (1), (3) and (4).

$$a_{n+1,n+1} = a_{n,n} + \frac{1}{2}(a_{n+1,n} + a_{n+1,n+2})$$

$$= \frac{1}{2} \sum_{i=1}^{n} \binom{n-1}{i-1} (a_{i,1} + a_{i,2i-1}) + \frac{1}{2}(a_{n+1,n} + a_{n+1,n+2})$$

$$= \frac{1}{2} \left(\sum_{i=1}^{n} \binom{n-1}{i-1} a_{i,1} + \sum_{i=1}^{n} \binom{n-1}{i-1} a_{i+1,1} \right)$$

$$+ \frac{1}{2} \left(\sum_{i=1}^{n} \binom{n-1}{i-1} a_{i,2i-1} + \sum_{i=1}^{n} \binom{n-1}{i-1} a_{i+1,2i+1} \right)$$

$$= \frac{1}{2}\left(\binom{n-1}{0} a_{1,1} + \sum_{i=2}^{n}\binom{n-1}{i-1} a_{i,1} + \sum_{i=1}^{n-1}\binom{n-1}{i-1} a_{i+1,1} + \binom{n-1}{n-1} a_{n+1,1}\right)$$

$$+\frac{1}{2}\left(\binom{n-1}{0} a_{1,1} + \sum_{i=2}^{n}\binom{n-1}{i-1} a_{i,2i-1} + \sum_{i=1}^{n-1}\binom{n-1}{i-1} a_{i+1,2i+1} + \binom{n-1}{n-1} a_{n+1,2n+1}\right)$$

$$= \frac{1}{2}\left(\binom{n}{0} a_{1,1} + \sum_{i=2}^{n}\binom{n}{i-1} a_{i,1} + \binom{n}{n} a_{n+1,1}\right)$$

$$+\frac{1}{2}\left(\binom{n}{0} a_{1,1} + \sum_{i=2}^{n}\binom{n}{i-1} a_{i,2i+1} + \binom{n}{n} a_{n+1,2n+1}\right)$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n+1}\binom{n}{i-1} a_{i,1} + \sum_{i=1}^{n+1}\binom{n}{i-1} a_{i,2i-1}\right)$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n+1}\binom{n}{i-1} (a_{i,1} + a_{i,2i-1}\right).$$

That proves the Theorem. □

In some cases, it is suitable for the generating sequences to be $\{b_i\}_{i\geq 1}$ and $\{c_i\}_{i\geq 1}$. Then the generated sequence has the form $\{a_i\}_{i\geq 1}$, where its n-th member has the form

$$a_n = \frac{1}{2}\sum_{i=1}^{n}\binom{n-1}{i-1}(b_i + c_i).$$

Let us finish with two partial cases. If the generating sequences are the arithmetic progressions "$a, a+2b, a+4b, ...$" and "$a, a+2c, a+4c, ...$". Then the triangle has the form

$$\begin{array}{ccccccc}
 & & & a & & & \\
 & & a+2b & 2a+b+c & a+2c & & \\
 & a+4b & 2a+6b & 4a+4b+4c & 2a+6c & a+4c & \\
a+6b & 2a+10b & 4a+16b & 8a+12b+12c & 4a+16c & 2a+10c & a+6c \\
 & \vdots & & \vdots & & \vdots &
\end{array}$$

Therefore, the n-th member of the generated sequence for the natural number $n \geq 1$ is

$$\alpha_n = 2^{n-1}a + (n-1)2^{n-2}(b+c).$$

When we like to receive an arithmetic progression as a generated sequence of a triangle, then the Pascal's like triangle can have, for example, the form:

$$\begin{array}{ccccccccccc}
 & & & & & a & & & & & \\
 & & & & b-1 & a+b & b+1 & & & & \\
 & & & 0 & b-1 & a+b & b+1 & 0 & & & \\
 & & 0 & 0 & b-1 & a+b & b+1 & 0 & 0 & & \\
 & 0 & 0 & 0 & b-1 & a+b & b+1 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & b-1 & a+b & b+1 & 0 & 0 & 0 & 0 \\
 & & & \vdots & & \vdots & & \vdots & & &
\end{array}$$

In future, three dimensional forms of these Pascal's like triangles will be discussed.

References

[1] Atanassov, K., On some Pascal's like triangles. Part 1. *Notes on Number Theory and Discrete Mathematics*, Vol. 13, 2007, No. 1, 31–36.

[2] Atanassov, K., On some Pascal's like triangles. Part 2. *Notes on Number Theory and Discrete Mathematics*, Vol. 13, 2007, No. 2, 10–14.

[3] Atanassov, K., On some Pascal's like triangles. Part 3. *Notes on Number Theory and Discrete Mathematics*, Vol. 13, 2007, No. 3, 20–25.

[4] Atanassov, K., On some Pascal's like triangles. Part 4. *Notes on Number Theory and Discrete Mathematics*, Vol. 13, 2007, No. 4, 11–20.

[5] Atanassov, K., On some Pascal's like triangles. Part 5. *Advanced Studies in Contemporary Mathematics*, Vol. 21, 2011, No. 3, 291–299.

[6] Atanassov, K., On some Pascal's like triangles. Part 6. *Notes on Number Theory and Discrete Mathematics*, Vol. 20, 2014, No. 4, 40–46.

[7] Bondarenko, B., *Generalized Pascal's Triangles and Pyramids – Their Fractals, Graphs and Applications*, Tashkent, Fan, 1990 (in Russian).

[8] Čerin, Z., Sums of squares and products of Jacobsthal numbers. *Journal of Integer Sequences*, Vol. 10, 2007, Article 07.2.5.

[9] Goldwasser, J., W. Klostermeyer, M. Mays, G. Trapp, The density of ones in pascal's rhombus. *Discrete mathematics*, Vol. 204, 1999, 231–236.

[10] Leyendekkers, J., A. Shannon, J. Rybak. *Pattern Recognition: Modular Rings & Integer Structure*. Raffles KvB Monograph, No. 9, North Sydney, 2007.