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Three identities concerning Fibonacci and Lucas numbers

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Abstract: In the literature, there are many identities about Fibonacci and Lucas numbers. In this study, we give three identities concerning Fibonacci and Lucas numbers. Then we present some Diophantine equations such as $z^2 + x^2 + y^2 = xyz + 4$.

Keywords: Fibonacci numbers, Lucas numbers, Fibonacci matrix, Diophantine equation.
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1 Introduction

The Fibonacci sequence (F_n) is defined by $F_0 = 0$, $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. F_n is called the nth Fibonacci number. Fibonacci numbers for negative subscripts are defined as $F_{-n} = (-1)^{n+1} F_n$ for $n \geq 1$. It is well known that $F_{n+1} = F_n + F_{n-1}$ for every $n \in \mathbb{Z}$. The Lucas sequence (L_n) is defined as $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. L_n is called the nth Lucas number. Lucas numbers for negative subscripts are defined as $L_{-n} = (-1)^n L_n$. It can be seen that $L_n = F_{n+1} + F_{n-1}$ for every $n \in \mathbb{Z}$. It is well known that $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ and $L_n = \alpha^n + \beta^n$ for every $n \in \mathbb{Z}$ where $\alpha = \left(1 + \sqrt{5}\right)/2$ and $\beta = \left(1 - \sqrt{5}\right)/2$ are the roots of the polynomial $x^2 - x - 1$. These are known as Binet's Formula. It is well known that for the matrix $Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, we have $Q^2 = Q + I$ and $Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$. Moreover, there are two well known identities given by

$$L_n^2 - 5F_n^2 = 4(-1)^n \tag{1}$$

and

$$\alpha^n = \alpha F_n + F_{n-1} \tag{2}$$

for every $n \in \mathbb{Z}$. Many identities for Fibonacci and Lucas numbers are proved by using Binet's Formula, induction or Fibonacci matrices (see [6, 8]). Here, we can give some of them as follows:

$$\begin{array}{rcll}
F_{n+k}F_{n-k} - F_n^2 & = & (-1)^{n+k+1} F_k^2 & \text{(Catalan identity)} \\
F_m F_{n+1} - F_n F_{m+1} & = & (-1)^n F_{m-n} & \text{(d'Ocagne's identity)} \\
F_{n-1}F_{n+1} - F_n^2 & = & (-1)^n & \text{(Cassini identity)} \\
F_{k-1}F_n + F_k F_{n+1} & = & F_{n+k} & \text{(Honsberger identity)}
\end{array}$$

In this study, we give some new identities concerning Fibonacci and Lucas numbers. I think the identities we present here will be found fascinating. Then we present some Diophantine equations such as $z^2 + x^2 + y^2 = xyz + 4$For a history of the Fibonacci numbers we refer the reader to the reference [1]. For more information about Fibonacci and Lucas numbers one can consult [6], [8], and [2].

2 Main theorems

The following lemma and theorem are given in [5].

Lemma 1. *If X is a square matrix with $X^2 = X + I$, then $X^n = F_n X + F_{n-1} I$ for all $n \in \mathbb{Z}$.*

Theorem 1. *Let X be an arbitrary 2×2 matrix. Then $X^2 = X + I$ if and only if X is of the form*

$$X = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} \text{ with } \det X = -1 \text{ where } b \neq 0 \text{ or } c \neq 0,$$

or

$$X = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \text{ where } a, b \in \{\alpha, \beta\}.$$

Theorem 2. *Let $A = \begin{bmatrix} \alpha & 0 \\ 1 & \beta \end{bmatrix}$. Then*

$$A^n = \begin{bmatrix} \alpha^n & 0 \\ F_n & \beta^n \end{bmatrix}$$

Proof. Since $A^2 = A + I$, by Lemma 1 and (2) it follows that

$$A^n = F_n A + F_{n-1} I = \begin{bmatrix} \alpha F_n + F_{n-1} & 0 \\ F_n & \beta F_n + F_{n-1} \end{bmatrix} = \begin{bmatrix} \alpha^n & 0 \\ F_n & \beta^n \end{bmatrix}.$$

By using the above theorem, we give the following theorem.

Theorem 3.

$$(-1)^{n+m} L_{n+m}^2 - 5(-1)^n F_n^2 - 5(-1)^m F_m^2 = 5(-1)^{n+m} F_n F_m L_{n+m} + 4$$

for all $n, m \in \mathbb{Z}$*.*

Proof. It can be seen that

$$A^{n+1} + A^{n-1} = \begin{bmatrix} \sqrt{5}\alpha^n & 0 \\ L_n & -\sqrt{5}\beta^n \end{bmatrix}.$$

If we use matrix multiplication $(A^{n+1}+A^{n-1})(A^{m+1}+A^{m-1})$, we get $\sqrt{5}F_{n+m} = L_n\alpha^m - L_m\beta^n$. Thus

$$\begin{aligned} 5F_{n+m}^2 &= \left(\sqrt{5}F_{n+m}\right)\left(\sqrt{5}F_{n+m}\right) = (L_n\alpha^m - L_m\beta^n)(L_m\alpha^n - L_n\beta^m) \\ &= L_m L_n L_{n+m} - (-1)^n L_m^2 - (-1)^m L_n^2. \end{aligned}$$

Since $5F_{n+m}^2 = L_{n+m}^2 - 4(-1)^{n+m}$ by (1), it follows that

$$(-1)^{n+m} L_n L_m L_{n+m} + 4 = (-1)^{n+m} L_{n+m}^2 + (-1)^n L_n^2 + (-1)^m L_m^2$$

This completes the proof. □

Theorem 4.

$$5(-1)^{n+m} F_{n+m}^2 - (-1)^n L_n^2 + 5(-1)^m F_m^2 = 5(-1)^{n+m} L_n F_m F_{n+m} - 4$$

for all $n, m \in \mathbb{Z}$*.*

Proof. By using a similar argument, it can be shown that $L_{n+m} = \alpha^m L_n - \sqrt{5}\beta^n F_m$ and $L_{n+m} = \sqrt{5}\alpha^n F_m + \beta^m L_n$. Therefore, we have

$$\begin{aligned} L_{n+m}^2 &= \left(\alpha^m L_n - \sqrt{5}\beta^n F_m\right)\left(\sqrt{5}\alpha^n F_m + \beta^m L_n\right) \\ &= 5F_m L_n F_{n+m} + (-1)^m L_n^2 + 5(-1)^{n+1} F_m^2. \end{aligned}$$

Since $L_{n+m}^2 = 5F_{n+m}^2 + 4(-1)^{n+m}$, by (1), it is seen that

$$5(-1)^{n+m} F_{n+m}^2 - (-1)^n L_n^2 + 5(-1)^m F_m^2 = 5(-1)^{n+m} L_n F_m F_{n+m} - 4$$

This completes the proof. □

From the above theorems, we can give the following corollaries.

Corollary 1. *If n and m are even integers, then $(x,y,z) = (F_n, F_m, L_{n+m})$ is a solution of the equation $z^2 - 5x^2 - 5y^2 = 5xyz + 4$. If n and m are odd integers, then $(x,y,z) = (F_n, F_m, L_{n+m})$ is a solution of the equation $z^2 + 5x^2 + 5y^2 = 5xyz + 4$, and if n is an odd integer and m is an even integer, then $(x,y,z) = (F_n, F_m, L_{n+m})$ is a solution of the equation $z^2 - 5x^2 + 5y^2 = 5xyz - 4$.*

Corollary 2. *If n and m are even integers, then $(x,y,z) = (L_n, L_m, L_{n+m})$ is a solution of the equation $z^2 + x^2 + y^2 = xyz + 4$, if n and m are odd integers, then $(x,y,z) = (L_n, L_m, L_{n+m})$ is a solution of the equation $z^2 - x^2 - y^2 = xyz + 4$. Moreover, if m is an odd integer and n is an even integer, then $(x,y,z) = (L_n, L_m, L_{n+m})$ is a solution of the equation $z^2 - x^2 + y^2 = xyz - 4$.*

Corollary 3. *If n and m are even integers, then $(x,y,z) = (L_n, F_m, F_{n+m})$ is a solution of the equation $5z^2 - x^2 + 5y^2 = 5xyz - 4$.*

3 Concluding remark

There are many identities concerning Fibonacci and Lucas numbers in the literature. Moreover, solutions of some Diophantine equations can be given in terms of Fibonacci and Lucas sequences. We can state some of these equations as follows (see, for example, [5, 4]):

All positive integer solutions of the equation $x^2 - 5y^2 = 4$ are given by $(x, y) = (L_{2n}, F_{2n})$ with $n \geq 1$.

All positive integer solutions of the equation $x^2 - 5y^2 = -4$ are given by $(x, y) = (L_{2n-1}, F_{2n-1})$ with $n \geq 1$.

All positive integer solutions of the equation $x^2 - xy - y^2 = 1$ are given by $(x, y) = (F_{2n+1}, F_{2n})$ with $n \geq 1$.

All positive integer solutions of the equation $x^2 - xy - y^2 = -1$ are given by $(x, y) = (F_{2n}, F_{2n-1})$ with $n \geq 1$.

All positive integer solutions of the equation $x^2 - xy - y^2 = 5$ are given by $(x, y) = (L_{2n}, L_{2n-1})$ with $n \geq 1$.

All positive integer solutions of the equation $x^2 - xy - y^2 = -5$ are given by $(x, y) = (L_{2n+1}, L_{2n})$ with $n \geq 0$.

Although, $(x, y, z) = (L_{2n}, L_{2m}, L_{2n+2m})$ is a positive solution of the equation $z^2 + x^2 + y^2 = xyz + 4$, if a is an integer greater then 1, then $(x, y, z) = (a^2 - 2, a, a)$ is a positive integer solution of the same equation. Many mathematicians are interested in solving Diophantine equations. I think it is a bit difficult and interesting to give all integer (positive integer) solutions of the Diophantine equations

$$\begin{aligned}
z^2 - 5x^2 - 5y^2 &= 5xyz + 4, \\
z^2 + 5x^2 + 5y^2 &= 5xyz + 4, \\
z^2 - 5x^2 + 5y^2 &= 5xyz - 4, \\
z^2 + x^2 + y^2 &= xyz + 4, \\
z^2 - x^2 - y^2 &= xyz + 4, \\
z^2 - x^2 + y^2 &= xyz - 4,
\end{aligned}$$

and

$$5z^2 + 5y^2 - x^2 = 5xyz - 4,$$

although they have infinite many integer solutions by the above corollaries.

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