Notes on Number Theory and Discrete Mathematics
ISSN 1310–5132
Vol. 20, 2014, No. 5, 35–39

A combinatorial proof of multiple angle formulas involving Fibonacci and Lucas numbers

Fernando Córes[1] and Diego Marques[2]

[1] Department of Mathematics, University of Brasilia
Brasilia, DF, Brazil
e-mail: fccores@gmail.com

[2] Department of Mathematics, University of Brasilia
Brasilia, DF, Brazil
e-mail: diego@mat.unb.br

Abstract: Let F_n and L_n be the nth Fibonacci and Lucas number, respectively. In this note, we give a combinatorial proof for the following identity

$$F_{kn+p} = F_n^k F_p + \sum_{i=1}^{k} F_n^{i-1}(F_{n-1}F_{(k-i)n+p} + F_n F_{(k-i)n+p-1}).$$

Keywords: Fibonacci, Lucas, Multiple angle, Combinatorial proof.
AMS Classification: 11B39.

1 Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. A few terms of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, \ldots.$$

We cannot go very far in the lore of Fibonacci numbers without encountering its companion Lucas sequence $(L_n)_{n\geq 0}$, which follows the same recursive pattern as the Fibonacci numbers, but with initial values $L_0 = 2$ and $L_1 = 1$:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, \ldots.$$

The Fibonacci numbers are well-known for possessing wonderful and amazing properties (consult [1] together with its very extensive annotated bibliography for additional references and history). In 1963, the Fibonacci Association was created to provide enthusiasts an opportunity to share ideas about these intriguing numbers and their applications. Also, in the issues of *The Fibonacci Quarterly* we can find many new facts, applications, and relationships about Fibonacci numbers.

The search for identities involving Fibonacci numbers has always been a popular area of research. Among the several identities, we are interested in *multiple angle* formulas. The most famous among them is the very useful identity $F_{2n} = F_n L_n$. This identity can be easily rewritten as $F_{2n} = F_n^2 + 2F_n F_{n-1}$ (recently, this identity appeared as Theorem 2.5 of [2]). For the Lucas case, we have $L_{2n} = (5F_n^2 + L_n^2)/2$. Below we present some general multiple angle formulas

$$\begin{aligned}
F_{kn} &= L_k F_{k(n-1)} - (-1)^k F_{k(n-2)} \\
&= \frac{1}{2^{k-1}} \cdot \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2i+1} 5^i F_n^{2i+1} L_n^{k-1-2i} \\
&= F_n \cdot \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \binom{k-1-i}{i} (-1)^{i(n+1)} L_n^{k-1-2i} \\
&= \sum_{i=0}^{k} \binom{k}{i} F_i F_n^i F_{n-1}^{k-i} \\
&= \sum_{i=0}^{k} \binom{k}{i} F_{-i} F_n^i F_{n+1}^{k-i}
\end{aligned}$$

and

$$\begin{aligned}
L_{kn} &= L_k L_{k(n-1)} - (-1)^k L_{k(n-2)} \\
&= \frac{1}{2^{k-1}} \cdot \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} 5^i F_n^{2i} L_n^{k-2i} \\
&= \sum_{i=0}^{k} \binom{k}{i} L_i F_n^i F_{n-1}^{k-i}.
\end{aligned}$$

We also have the general formula

$$F_{kn+p} = \sum_{i=0}^{k} \binom{k}{i} F_{p-i} F_n^i F_{n+1}^{k-i}.$$

In this note, we shall give combinatorial proofs for "new" (to the best of authors' knowledge) multiple angle formulas. More precisely, we have the following

Theorem 1.1. *For all positive integers n, k and p, it holds that*

$$F_{kn+p} = F_n^k F_p + \sum_{i=1}^{k} F_n^{i-1} (F_{n-1} F_{(k-i)n+p} + F_n F_{(k-i)n+p-1}). \tag{1}$$

As application, we prove that

Corollary 1.2. *For all positive integers n, k and p, it holds that*

$$L_{kn+p} = F_n^k L_p + \sum_{i=1}^{k} F_n^{i-1}(F_{n-1}L_{(k-i)n+p} + F_n L_{(k-i)n+p-1}). \tag{2}$$

2 The proof of the theorem

2.1 Auxiliary facts

Before the proof, for the convenience of the reader, we shall recall combinatorial interpretations for Fibonacci and Lucas numbers.

By a *binary* sequence, we call a finite *word* written using only 0's and 1's. A binary sequence is said to be of *type* 0, if its first and last terms are equal to 0 and such that it does not contain two consecutive 1's.

Lemma 2.1. *The number of binary words of type* 0 *with length n is equal to F_n.*

Proof. Let p_n be the number of binary words of type 0 with length n. Clearly, $p_1 = p_2 = 1$ ($\{0, 00\}$). Also, such a binary word of type 0 either starts with 01 or 00. In the first case, it remains $01\underbrace{0\ldots0}_{n-2 \text{ bits}}$ having, by definition, p_{n-2} words. In the second case, we have $1\underbrace{0\ldots0}_{n-1 \text{ bits}}$ which are counted in p_{n-1} ways. In conclusion, $p_n = p_{n-1} + p_{n-2}$ which completes the proof. □

A binary sequence is said to be of *type* 1 if there is no 1's simultaneously at first and last positions.

Lemma 2.2. *The number of binary words of type* 1 *with length n is equal to L_n.*

Proof. Let c_n be the number of such words with length n. Clearly, $c_1 = 1$ ($\{0\}$) and $c_2 = 3$ ($\{00, 01, 10\}$). Also, such a word either starts with 1 or 0. In the first case, the second and the last bit are 0, so by the previous lemma, we have F_{n-1} such words. In the case of starting with 0, the last term can be 0 and so we have F_n words, or it ends with 1 and so we obtain F_{n-1} words. In conclusion, $c_n = F_{n-1} + F_n + F_{n-1} = F_{n-1} + F_{n+1}$. A straightforward calculation yields $c_{n-1} + c_n = c_{n+1}$ which completes the proof. □

As a consequence, we have that

Lemma 2.3. *For all positive integer n, it holds that $L_n = F_{n-1} + F_{n+1}$.*

Now, we are ready to deal with the proof of theorem.

2.2 The proof of Theorem 1.1

In order to simplify our argument, we shall rewrite (1) as

$$F_p F_n^k = F_{kn+p} - \sum_{i=1}^{k} F_n^{i-1}(F_{n-1}F_{(k-i)n+p} + F_n F_{(k-i)n+p-1}).$$

The left-hand side above counts the number of juxtapositions of k binary sequences of length n and one sequence of length p, all of type 0, which is $F_p F_n^k$ (by Lemma 2.1). On the other hand, we can count these juxtapositions excluding from the F_{kn+p} binary sequences of length n and type 0 those ones having the *bits* 0 and 1 or 1 and 0 at positions in and $in+1$, for $i \in \{1, \dots, k\}$, respectively. In the case of a sequence having 0 and 1 at positions in and $in+1$, respectively (see below)

$$\underbrace{0\dots0}_{in \text{ bits}}1 \quad \underbrace{0\dots0}_{(k-i)n+p-1 \text{ bits}}$$

we have F_n^i juxtapositions of the first in block and $F_{(k-i)n+p-1}$ for the last block (since 1 is fixed at position $in+1$). Thus, there exist $F_n^i F_{(k-i)n+p-1}$ juxtapositions. In the second case, that is, a sequence having 1 and 0 at positions in and $in+1$, respectively (see below)

$$\underbrace{0\dots0}_{(i-1)n \text{ bits}}\ \underbrace{0\dots0}_{n-1 \text{ bits}}1 \quad \underbrace{0\dots0}_{(k-i)n+p \text{ bits}}$$

we obtain F_n^{i-1} for the first block, F_{n-1} for the second one and $F_{(k-i)n+p}$ for the last one. In a total of $F_n^{i-1}F_{n-1}F_{(k-i)n+p}$ ways. In conclusion we have

$$\sum_{i=1}^{k}(F_n^i F_{(k-i)n+p-1} + F_n^{i-1}F_{n-1}F_{(k-i)n+p})$$

excluded sequences and therefore

$$F_p F_n^k = F_{kn+p} - \sum_{i=1}^{k} F_n^{i-1}(F_{n-1}F_{(k-i)n+p} + F_n F_{(k-i)n+p-1})$$

holds. □

2.3 The proof of Corollary 1.2

By Lemma 2.3, we have that $L_{kn+p} = F_{kn+p-1} + F_{kn+p+1}$. Thus, by Theorem 1.1,

$$\begin{aligned} L_{kn+p} &= F_{kn+p-1} + F_{kn+p+1} \\ &= F_n^k F_{p-1} + \sum_{i=1}^{k} F_n^{i-1}(F_{n-1}F_{(k-i)n+p-1} + F_n F_{(k-i)n+p-2}) \\ &\quad + F_n^k F_{p+1} + \sum_{i=1}^{k} F_n^{i-1}(F_{n-1}F_{(k-i)n+p+1} + F_n F_{(k-i)n+p}) \end{aligned}$$

$$
\begin{aligned}
&= F_n^k(F_{p-1}+F_{p+1}) + \sum_{i=1}^{k} F_n^{i-1}(F_{n-1}(F_{(k-i)n+p-1}+F_{(k-i)n+p+1}) \\
&\quad +F_n(F_{(k-i)n+p-2}+F_{(k-i)n+p})) \\
&= F_n^k L_p + \sum_{i=1}^{k} F_n^{i-1}(F_{n-1}L_{(k-i)n+p}+F_nL_{(k-i)n+p-1})
\end{aligned}
$$

The proof is then complete. □

Acknowledgements

The authors are grateful to the referees for reading this paper. During the preparation of this work, the first author was supported by CAPES (Coordenaçao de Aperfeiçoamento de Pessoal de Nível Superior). The second author is supported by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) and FAP-DF (Fundaçao de Apoio à Pesquisa do Distrito Federal).

References

[1] Koshy, T., *Fibonacci and Lucas Numbers with Applications*, Wiley, New York, 2001.

[2] Thongmoon, M., New identities for the even and odd Fibonacci and Lucas Numbers, *Int. J. Contemp. Math. Sciences*,Vol. 4, 2009, 671–676.