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Extensions to the Zeckendorf Triangle

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Abstract: This note extends some of the characteristics of a Zeckendorf triangle composed of Fibonacci number multiples of the Fibonacci sequence.
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1 Introduction

This note extends a result in [6] which built on work by Griffiths [2] on a form of the Zeckendorf Triangle. A left-corrected form appears in Table 1 where it can be seen that the columns, $\{F_{m,n}\}$, are Fibonacci number multiples, $F_{m,n} = F_{m-n+1}F_n$, $m > n$, of the numbers in the Fibonacci sequence, in which m and n designate rows and columns respectively

1										
1	1									
2	1	2								
3	2	2	3							
5	3	4	3	5						
8	5	6	6	5	8					
13	8	10	9	10	8	13				
21	13	16	15	15	16	13	21			
34	21	26	24	25	24	26	21	34		
55	34	42	39	40	40	39	42	34	55	
89	55	68	63	65	64	65	63	68	55	89

Table 1. A form of the Zeckendorf Triangle

The column sequences are actually particular cases of the generalized Fibonacci and Lucas sequences $\{F_{m,n}\}$, which satisfy the Fibonacci partial recurrence relation [1]

$$F_{m,n} = F_{m,n-1} + F_{m,n-2},\ m \geq 0,\ n > 2.$$

2 Sequences within the triangle

We now label the sequences of diagonal, row and partial column sums by $\{d_n\}$, $\{r_n\}$, $\{c_n\}$, respectively. We observe the sequences so generated in Table 2.

n	1	2	3	4	5	6	7	8	9	10	11
$\{d_n\}$	1	1	3	4	9	13	25	38	68	106	182
$\{r_n\}$	1	2	5	10	20	38	71	130	235	420	744
$\{c_n\}$	1	2	6	15	40	104	273	714	1870	4895	12816
$\{b_n\}$	1	1	4	9	25	64	169	441	1156	3025	7921

Table 2. Sequences within the Zeckendorf Triangle

The $\{b_n\}$ sequence has been formed from the central column of the original isosceles form of the triangle in [2], as in Table 3.

1

1 1

2 1 2

3 2 2 3

5 3 4 3 5

8 5 6 6 5 8

13 8 10 9 10 8 13

Table 3. Isosceles form of the Zeckendorf triangle

Thus,

$$\{b_n\} \equiv \{z_{1,1}, z_{3,2}, z_{5,3}, z_{7,4}, z_{9.5}, \ldots\} \equiv \{1, 1, 4, 9, 25, 64, \ldots\} \qquad (2.1)$$

in which the $\{z_{i,j}\}$ are the elements of the isosceles form of the Zeckendorf triangle.

The $\{c_n\}$ sequence is formed from the cumulative partial sums of $\{b_n\}$:

$$\begin{aligned} b_n &= c_n - c_{n-1}, n > 1, \\ &= 2F_{2n-2} - b_{n-3},\ n > 3 \\ &= F_n^2. \end{aligned}$$

That is, by the repeated application of the first of these (with c_0 set to zero) we find that

$$c_n = \sum_{j=1}^{n} F_n^2$$
$$= F_n F_{n+1}$$

which can be confirmed from the table. We also observe too that, for $n \geq 2$,

$$d_{2n} = d_{2n-1} + d_{2n-2}\,,$$

and

$$d_{2n+1} = d_{2n} + d_{2n-1} + F_{n+1}$$

Then, from the triangle, it can be seen that we get the recurrence relations

$$d_{2n-j} + d_{2n-j+1} + \delta_{1,j} F_n = d_{2n-j+2},\ j = 0,1, \qquad (2.2)$$

in which $\delta_{i,j}$ is the Kronecker delta and $\{F_n\}$ are the Fibonacci numbers. Similarly, the $\{r_n\}$ is a Fibonacci convolution sequence [5] where

$$5r_n = nF_{n+2} + (n+2)F_n \qquad (2.3)$$

and

$$r_n + r_{n+1} + F_{n+2} = r_{n+2} \qquad (2.4)$$

We note that (2.4) reduces to (2.3) with repeated use of the Fibonacci recurrence relation. The row numbers, $\{r_n\}$, were shown by Griffiths [3] to be convolutions of the Fibonacci numbers.

3 Concluding extension

We can obtain another connection between the Fibonacci numbers and the Zeckendorf representations of the integers by defining another partial column sequence in Table 1, namely $\{k_{m,n}\}$, in which m identifies the row and n identifies the column as in Table 4.

$m\rightarrow$ $n\downarrow$	1	2	3	4	5	6
1	1					
2	2	1				
3	4	2	2			
4	7	4	4	3		
5	12	7	8	6	5	
6	20	12	14	12	10	8

Table 4. Partial column sums from Table 1

It can then be established that

$$k_{m,n} = k_{m-1,n-1} + k_{m-2,n-2},\ m > 2, n > 2, \qquad (3.1)$$

and within the columns

$$k_{m,n} = k_{m-1,n} + k_{m-1,} + F_n,\ n > m+1. \tag{3.2}$$

Many similar enumeration themes are unified in the Riordan Group [7]. Connections within the rows are left to the interested reader [3]. Connections within the columns are related to the leading diagonals in Hoggatt's trimmed Pascal triangles [4].

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