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On upper Hermite–Hadamard inequalities for geometric-convex and log-convex functions

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Abstract: We offer connections between upper Hermite–Hadarmard type inequalities for geometric convex and logarithmically convex functions.
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1 Introduction

Let $I \subset \mathbb{R}$ be a nonvoid interval. A function $f : I \to (0, +\infty)$ is called **log-convex** (or logarithmically convex), if the function $g : I \to \mathbb{R}$, defined by $g(x) = \ln f(x)$, $x \in I$ is convex; i.e. satisfies

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y) \tag{1.1}$$

for all $x, y \in I$, $\lambda \in [0, 1]$.

Inequality (1) may be rewritten for the function f, as

$$f(\lambda x + (1-\lambda)y) \leq (f(x))^{\lambda}(f(y))^{1-\lambda}, \tag{1.2}$$

for $x, y \in I$, $\lambda \in [0, 1]$.

If one replaces the weighted arithmetic mean $\lambda x + (1-\lambda)y$ of x and y with the weighted geometric mean, i.e. $x^{\lambda}y^{1-\lambda}$, then we get the concept of **geometric-convex** function $f : I \subset (0, +\infty) \to (0, +\infty)$

$$f(x^{\lambda}y^{1-\lambda}) \leq (f(x))^{\lambda}(f(y))^{1-\lambda}, \tag{1.3}$$

for $x, y \in I$, $\lambda \in [0, 1]$.

These definitions are well-known in the literature, we quote e.g. [7] for an older and [4] for a recent monograph on this subject.

Also, the well-known Hermite–Hadamard inequalities state that for a convex function g of (1.1) one has

$$g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b g(x)dx \le \frac{g(a)+g(b)}{2}, \tag{1.4}$$

for any $a, b \in I$.

We will call the right side of (4) as the **upper Hermite–Hadamard inequality**.

By applying the weighted geometric mean-arithmetic mean inequality

$$a^{\lambda}b^{1-\lambda} \le \lambda a + (1-\lambda)b, \tag{1.5}$$

the following properties easily follow:

Lemma 1. *(i) If $f : I \to (0,\infty)$ is log-convex, then it is convex;*

(ii) If $f : I \subset (0,\infty) \to (0,\infty)$ is increasing and log-convex, then it is geometric convex.

Proof. We offer for sake of completeness, the simple proof of this lemma.

(i) One has by (1.2) and (1.5):

$$f(\lambda x + (1-\lambda)y) \le (f(x))^{\lambda}(f(y))^{1-\lambda} \le \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in I$, $\lambda \in [0,1]$.

(ii) $f(x^{\lambda}y^{1-\lambda}) \le f(\lambda x + (1-\lambda)y)$ by (1.5) and the monotonicity of f. Now, by (1.2) we get (1.3).

Let $L(a,b)$ denote the logarithmic mean of two positive real numbers and b, i.e.

$$L(a,b) = \frac{b-a}{\ln b - \ln a} \text{ for } a \ne b;\ L(a,a) = a. \tag{1.6}$$

In 1997, Gill, Pearce and Pečarić [1] have proved the following upper Hermite–Hadamard type inequality:

Theorem 1.1. *If $f : [a,b] \to (0,+\infty)$ is log-convex, then*

$$\frac{1}{b-a}\int_a^b f(x)dx \le L(f(a), f(b)), \tag{1.7}$$

where L is defined by (1.6).

Recently, Xi and Qi [6] proved the following result:

Theorem 1.2. *Let $a, b > 0$ and $f : [a,b] \to (0,+\infty)$ be increasing and log-convex. Then*

$$\frac{1}{\ln b - \ln a}\int_a^b f(x)dx \le L(af(a), bf(b)). \tag{1.8}$$

Prior to [6], Iscan [2] published the following result:

Theorem 1.3. *Let $a, b > 0$ and $f : [a,b] \to (0,\infty)$ be integrable and geometric-convex function. Then*

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \le L(f(a), f(b)). \tag{1.9}$$

In case of f increasing and log-convex, (1.9) is stated in [6], too. However, by Lemma 1(ii), clearly Theorem 1.3 is a stronger version.

In what follows, we shall offer refinements of (1.8) and (1.9). In fact, in almost all cases, inequality (1.7) is the strongest from the above.

2 Main results

First we prove that the result of Theorem 1.2 holds true in fact for geometric-convex functions:

Theorem 2.1. *Relation (1.8) holds true when f is integrable geometric convex function.*

Proof. First remark that when f is geometric-convex, the same is true for the function $g(x) = xf(x)$, $x \in I$. Indeed, one has

$$\begin{aligned} g(x^\lambda y^{1-\lambda}) &= x^\lambda y^{1-\lambda} f(x^\lambda y^{1-\lambda}) \le x^\lambda y^{1-\lambda} (f(x))^\lambda (f(y))^{1-\lambda} \\ &= (xf(x))^\lambda (yf(y))^{1-\lambda} = (g(x))^\lambda (g(y))^{1-\lambda}, \end{aligned}$$

for all $x, y \in I$, $\lambda \in [0,1]$. Therefore, by (1.3), g is geometric convex.

Apply now inequality (1.9) for $xf(x)$ in place of $f(x)$. Relation (1.8) follows.

In what follows, we shall need the following auxiliary result:

Lemma 2.1. *Suppose that $b > a > 0$ and $q \ge p > 0$. Then one has*

$$L(pa, qb) \ge L(p,q)L(a,b), \tag{2.1}$$

where L denotes the logarithmic mean, defined by (1.6).

Proof. Two proofs of this result may be found in [5]. Relation (2.1) holds true in a general setting of the Stolarksy means, see [3] (Theorem 3.8).

We offer here a proof of (2.1) for the sake of completeness. As

$$L(a,b) = \int_0^1 b^u a^{1-u} du, \tag{2.2}$$

applying the Chebysheff integral inequality

$$\frac{1}{y-x}\int_x^y f(t)dt \cdot \frac{1}{y-x}\int_x^y g(t)dt < \frac{1}{y-x}\int_x^y g(t)f(t)dt, \tag{2.3}$$

where $x < y$ and $f, g : [x,y] \to \mathbb{R}$ are strictly monotonic functions of the same type; to the particular case

$$[x,y] = [0,1];\ f(t) = b^t a^{1-t} = a\left(\frac{b}{a}\right)^t$$

and

$$g(t) = q^t p^{1-t} = p\left(\frac{q}{p}\right)^t$$

for $b > a$ and $q > p$; by (2.2), relation (2.1) follows. For $p = q$ one has equality in (2.1).

One of the main results of this paper is stated as follows:

Theorem 2.2. *Let $b > a > 0$ and suppose that $f : [a,b] \to \mathbb{R}$ is log-convex. Suppose that $f(b) \ge f(a)$. Then one has*

$$\frac{1}{b-a}\int_a^b f(x)dx \le L(f(a), f(b)) \le \frac{\ln b - \ln a}{b-a} \cdot L(af(a), bf(b)). \tag{2.4}$$

Proof. The first inequality of (2.4) holds true by Theorem 1.1.
Applying now Lemma 2.1, by $q = f(b) \geq f(a) = p$ and $b > a$, one has

$$L(f(a), f(b))L(a, b) \leq L(af(a), bf(b)).$$

As this is exactly the second inequality of (2.4), the proof of Theorem 2.2 is finished.

Remark 2.1. The weaker inequality of (2.4) is the result of Theorem 1.2, in an improved form (in place of increasing f, it is supposed only $f(b) \geq f(a)$).

When $f(b) > f(a)$, there is strict inequality in the right side of (2.4).

Theorem 2.3. *Let $b > a > 0$ and $f : [a, b] \to \mathbb{R}$ log-convex function. Suppose that $\frac{f(b)}{b} \geq \frac{f(a)}{a}$. Then one has*

$$\frac{1}{b-a}\int_a^b \frac{f(x)}{x}dx \leq L\left(\frac{f(a)}{a}, \frac{f(b)}{b}\right) \leq \frac{\ln b - \ln a}{b-a} \cdot L(f(a), f(b)). \tag{2.5}$$

Proof. First remark that $\frac{f(x)}{x}$ is log-convex function, too, being the product of the log-convex functions $\frac{1}{x}$ and $f(x)$. Thus, applying Theorem 1.1 for $\frac{f(x)}{x}$ in place of $f(x)$, we get the first inequality of (2.5).

The second inequality of (2.5) may be rewritten as

$$L\left(\frac{f(a)}{a}, \frac{f(b)}{b}\right) L(a, b) \leq L(f(a), f(b)),$$

and this is a consequence of Lemma 2.1 applied to $p = \frac{f(a)}{a}$, $q = \frac{f(b)}{b}$.

Remark 2.2. Inequality (2.5) offers a refinement of (1.9) whenever $\frac{f(b)}{b} \geq \frac{f(a)}{a}$. When here is strict inequality, the last inequality of (2.5) will be strict, too.

Lemma 2.2. *Suppose that $b > a > 0$ and $f : [a, b] \to \mathbb{R}$ is a real function such that $g(x) = \frac{f(x)}{x}$ is increasing in $[a, b]$. Then*

$$\int_a^b \frac{f(x)}{x}dx \leq \frac{1}{A}\int_a^b f(x)dx, \tag{2.6}$$

where $A = A(a, b) = \frac{a+b}{2}$ denotes the arithmetic mean of a and b.

Proof. Using Chebysheff's inequality (2.3) on $[x, y] = [a, b]$,

$$f(t) := \frac{f(t)}{t};\ g(t) := t,$$

which have the same type of monotonicity. Since

$$\frac{1}{b-a}\int_a^b t dt = \frac{a+b}{2} = A,$$

relation (2.6) follows.

The following theorem gives another refinement of (1.9):

Theorem 2.4. *Let $b > a > 0$ and $f : [a, b] \to \mathbb{R}$ log-convex, such that the function $x \mapsto \frac{f(x)}{x}$ is increasing on $[a, b]$. Then*

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \leq \frac{L}{A} \cdot L(f(a), f(b)) < L(f(a), f(b)), \tag{2.7}$$

where $L = L(a, b)$ denotes the logarithmic mean of a and b.

Proof. By (2.6) we can write

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \leq \left(\frac{b-a}{\ln b - \ln a} \right) \cdot \frac{1}{A} \cdot \left(\frac{1}{b-a} \int_a^b f(x) dx \right).$$

As $\frac{b-a}{\ln b - \ln a} = L$ and $\frac{1}{b-a} \int_a^b f(x) dx \leq L(f(a), f(b))$, by (1.7), the first inequality of (2.7) follows. The last inequality of (2.7) follows by the classical relation (see e.g. [3])

$$L < A. \tag{2.8}$$

Remark 2.2. As inequality (1.7) holds true with reversed sign of inequality, whenever f is log-concave (see [1]), (2.8) may be proved by an application for the log-concave function $f(x) = x$.

A counterpart to Lemma 2.1 is provided by:

Lemma 2.3. *If $\frac{q}{p} \geq \frac{b}{a} \geq 1$, then*

$$L(pa, qb) \leq L(p, q) A(a, b). \tag{2.9}$$

Proof. By letting $\frac{q}{p} = u$, $\frac{b}{a} = v$, inequality (2.9) may be rewritten as

$$\frac{uv - 1}{\ln(uv)} \leq \frac{u+1}{2} \cdot \frac{v-1}{\ln v}, \ u \geq v \geq 1. \tag{2.10}$$

If $v = 1$, then (2.9) is trivially satisfied, so suppose $v > 1$.

Consider the application

$$k(u) = (v-1)(u+1)\ln(uv) - 2(uv-1)\ln v, \ u \geq v.$$

One has $k(v) = 0$ and $k'(u) = (v-1)\left(\ln u + 1 + \frac{1}{u} \right) - (v+1)\ln v$.

Here $h(u) = \ln u + 1 + \frac{1}{u}$ has a derivative $h'(u) = \frac{u-1}{u^2} > 0$, so h is strictly increasing, implying $h(u) \geq h(v)$, One gets

$$k'(u) \geq (v-1)\left(\ln v + 1 + \frac{1}{v} \right) - (v+1)\ln v = \frac{v^2 - 1 - \ln(v^2)}{v} > 0,$$

on base of the classical inequality

$$\ln t \leq t - 1, \tag{2.11}$$

where equality occurs only when $t = 1$.

The function k being strictly increasing, we get $k(u) \geq k(v) = 0$, so inequality (2.9) follows.

Now, we will obtain a refinement of (1.9) for geometric convex functions:

Theorem 2.5. *Let $f : [a, b] \subset (0, \infty) \to (0, \infty)$ be a geometric convex function such that the application $x \mapsto \dfrac{f(x)}{x}$ is increasing. Then one has the inequalities*

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \leq \frac{1}{A(a,b)} \cdot L(af(a), bf(b)) \leq L(f(a), f(b)). \tag{2.12}$$

Proof. By Lemma 2.2 and Theorem 2.1, we can write

$$\begin{aligned} \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx &\leq \frac{1}{A(a,b)} \left(\frac{1}{\ln b - \ln a} \int_a^b f(x) dx \right) \\ &\leq \frac{L(af(a), bf(b))}{A(a,b)}. \end{aligned} \tag{2.13}$$

Now, applying Lemma 2.3 for $q = f(b)$, $p = f(a)$, by (2.9) we get

$$L(af(a), bf(b)) \leq L(f(a), f(b))A(a,b), \tag{2.14}$$

so the second inequality of (2.12) follows by the second inequality of (2.13).

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