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On integer solutions of $A^5 + B^3 = C^5 + D^3$

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Abstract: In this note, we study the diagonal nonhomogeneous symmetric Diophantine equation of the title, and show that when a solution has been found, a series of other solutions can be derived. This shows that difference of quintics equals difference of cubics for infinitely many integers. We do so using a method involving elliptic curves, which makes it possible to naturally find any solution in a matter of minutes.
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1 Introduction

The study of Diophantine equations has long been of interest to mathematicians. In this work, we consider

$$A^m + B^n = C^k + D^\ell, \tag{1}$$

where $m, n, k, \ell \in \mathbb{Z}_{>0}$. For arbitrary values of m, n, k, and ℓ, there does not seem to be many results on finding integral solutions to (1). From Dickson's History [3], it appears that the first, second, and third homogenous equations of the type (1), i.e., $A^n + B^n = C^n + D^n$ for $n = 2, 3, 4$, have been mainly investigated respectively by Pasternak, Binet, and Euler among others. The nonhomogeneous special cases of (1) that have been discussed are the classical ones proposed by Cunningham [3, pp. 698–699], in which $m = 5$, $n = 5$, $k = 2$, and $\ell = 2$, and by Gerardin [3, p. 566], in which $m = 3$, $n = 2$, $k = 3$, and $\ell = 2$. We note other specific cases have been also studied, see for example [1, 2].

In this short note, we study the particular equation of (1) with $m = 5$, $n = 3$, $k = 5$, and $\ell = 3$, namely

$$A^5 + B^3 = C^5 + D^3. \tag{2}$$

It is easy to see if (A_0, B_0, C_0, D_0) is a solution to the Diophantine equation (2), then so is $(\mu^3 A_0, \mu^5 B_0, \mu^3 C_0, \mu^5 D_0)$ for any rational μ. Searching for nontrivial integer solutions with $1 \leq A, B, C, D \leq 300$, $A < C$ found the solutions $(1, 32, 8, 1)$, $(1, 243, 27, 1)$, $(2, 10, 4, 2)$, $(3, 21, 6, 12)$, $(3, 220, 24, 139)$, $(6, 48, 9, 39)$, $(6, 298, 28, 210)$, $(7, 161, 14, 154)$, $(8, 243, 27, 32)$, $(9, 39, 6, 48)$, and $(28, 210, 6, 298)$. Computer searches can be used to find all solutions below a given bound, however it is challenging to find an infinite family of solutions.

Our main result is to find integral solutions to (2). We use a method, involving the theory of elliptic curves as a main tool, to find the solutions. In effect, we will show that to any possible suitable rational parameter t, it corresponds a rank-positive elliptic curve E_t so that each generator G_t of E_t leads to an integral solution to (2). Since E_t has an infinite number of rational solutions, this will yield an infinite number of integral solutions to (2). This shows implicitly that difference of quintics equals difference of cubics for infinitely many integers.

2 An infinite number of integral solutions to (2)

In this section, we compute solutions to (2) and show how to find solutions from generators on E_t.

Our method uses birational transformations to relate the equation (2) to a parametric positive rank elliptic curve E_t. For this, we introduce four new variables x, y, z, t, and set

$$A = -x + t, \ B = x + y, \ C = x + t, \ D = -x + y. \tag{3}$$

Substituting (3) into (2), and then brushing aside the uninteresting possibility $x = 0$, the equation (2) becomes

$$3y^2 = x^4 + (10t^2 - 1)x^2 + 5t^4, \tag{4}$$

or, equivalently,

$$E_t : Y^2 = 3X^4 + (270t^2 - 27)X^2 + 1215t^4, \tag{5}$$

where $Y = 27y$, $X = 3x$, and $t \neq 0$.

A short computer search reveals that (5) (or (4)) has a nontrivial solution $(A, B, C, D) = (-4, 80, 12, 64)$ for $t = 4$. In the sequel, we take $t = 4$. (One can pick up another value of t whose corresponding curve E_t has positive rank and get another class of integral solutions.) The curve (5) thus turns into

$$Y^2 = 3X^4 + 4293X^2 + 311040. \tag{6}$$

This quartic equation has rational point $(X, Y) = (24, 1944)$ among others. Put $Z = X - 24$. Hence, (6) becomes

$$Y^2 = 3Z^4 + 288Z^3 + 14661Z^2 + 371952Z + 3779136. \tag{7}$$

Using the following standard transformations given in Theorem 2.17 of [5]:

$$
\begin{aligned}
Z &= \frac{432(9U+49580)}{V},\\
Y &= \frac{24(-81V^2-85376760V-15498UV+1784880U^2+4916352800U+162U^3)}{V^2},
\end{aligned}
$$

the quartic (7) turns to the cubic

$$
V^2+\frac{574}{3}UV+1119744V=U^3+\frac{49580}{9}U^2-45349632U-249826083840, \qquad (8)
$$

which is a rank-two elliptic curve with generators $P_1=(-6660,16616)$ and $P_2=(-5724,41184)$ carried out with Sage software ([4]).

Now, completing the square on the left hand side of (8) yields

$$
E_4: W^2=U^3+14661U^2+61772544U+63630572544, \qquad (9)
$$

where $W=V+559872+(287/3)U$, and $G_1=(-6660,84348)$, $G_2=(-5724,53460)$ being generators.

For our next result, recall that the shape of Weierstrass curve requires that an element P be representable in the form $P=(u/v^2,w/v^3)$, where u,v,w are integers, with v coprime to uw.

Theorem 2.1. *Suppose $(u/v^2,w/u^3)$ is a rational point on the elliptic curve E_4, with $u,v,w\in\mathbb{Z}$, and $\gcd(v,uw)=1$. Let*

$$
\begin{cases}
A=4(-3w+1679616v^3+287uv)^{14}(3w+3675024v^3+685uv),\\
B=16(-3w+1679616v^3+287uv)^{23}(-36w^2-132uvw+10367154302976v^6\\
\qquad +6231739104uv^4+1638384wv^3+1234544u^2v^2+81u^3),\\
C=-12(-3w+1679616v^3+287uv)^{14}(3w+105264v^3+37uv),\\
D=16(-3w+1679616v^3+287uv)^{23}(-45w^2+132uvw+3684120v^3w\\
\qquad +12042913904640v^6+6852323736uv^4+1291657u^2v^2+81u^3).
\end{cases}
$$

Then (A,B,C,D) is an integral solution to $A^5+B^3=C^5+D^3$.

Proof. By aforementioned relations, we have

$$
\begin{aligned}
x &= -\frac{8(3W+199U+997704)}{-3W+287U+1679616},\\
y &= \frac{24}{(-3W+287U+1679616)^2}\{-27W^2+1774168W+54U^3+842067U^2\\
&\quad +4361354280U+7470022735872\}.
\end{aligned}
$$

Substituting in $U=u/v^2$ and $W=w/v^3$, we get

$$
\begin{aligned}
x &= -\frac{8(3w+997704v^3+199uv)}{-3w+1679616v^3+287uv},\\
y &= \frac{24}{(-3w+1679616v^3+287uv)^2}\{-27w^2+1774168wv^3+7470022735872v^6\\
&\quad +4361354280uv^4+842067u^2v^2+54u^3\}.
\end{aligned}
$$

Consequently,

$$\begin{cases} A = \dfrac{4(3w + 3675024v^3 + 685uv)}{-3w + 1679616v^3 + 287uv}, \\ B = \dfrac{16}{(-3w + 1679616v^3 + 287uv)^2}(-36w^2 - 132uvw + 1638384v^3 w \\ \quad +10367154302976v^6 + 6231739104uv^4 + 1234544u^2v^2 + 81u^3), \\ C = -12\dfrac{3w + 105264v^3 + 37uv}{-3w + 1679616v^3 + 287uv}, \\ D = \dfrac{16}{(-3w + 1679616v^3 + 287uv)^2}(-45w^2 + 132uvw + 3684120v^3 w \\ \quad +12042913904640v^6 + 6852323736uv^4 + 1291657u^2v^2 + 81u^3). \end{cases}$$

Using the fact that $(\mu^3 A, \mu^5 B, \mu^3 C, \mu^5 D)$ is a solution to (2) if (A, B, C, D) is, we can eliminate the denominators. The result follows immediately by choosing $\mu = -3w + 1679616v^3 + 287uv$. □

We note that the point $G_1 + G_2 = (-1188, 96228)$ on E_4 gives the solution $(A, B, C, D) = (12, 64, -4, 80)$ on (2) previously found. Computations show that this theorem yields also positive integer solutions. For example, the point $2G_1 + 2G_2$ on E_4 leads to

$$(89625865324, 22150923354823326352, 338012683980, 18626242614304094768)$$

on (2). The point $2G_1$ gives the solution

$$(A, B, C, D) = (3850788469738603004, 10325312124840891442240229608512, 7019980521187l5980, 12487708418886438298641426418928).$$

3 Conclusion

In this study, we used a method based on elliptic curves to find infinitely many integer solutions to the diagonal nonhomogeneous symmetric Diophantine equation (2), i.e., $A^5 + B^3 = C^5 + D^3$. In view of number theory, it means that difference of quintics equals difference of cubics for infinitely many integers. The equation (2) which we have focused on is in fact the $n = 3$ case of the more general equation

$$A^{2n-1} + B^n = C^{2n-1} + D^n. \tag{10}$$

If we let $A = -a + c$, $B = a + b$, $C = a + c$, and $D = -a + b$, then some algebra shows that when $n = 2$, the identity

$$(16c - 16a)^3 + (32a^2 + 64a + 96c^2)^2 = (16a + 16c)^3 + (32a^2 - 64a + 96c^2)^2,$$

gives an infinite parameterized family of integral solutions.

It is remarkable that finding integral solutions to (10) for $n \geq 4$ would be interesting. Preliminary computer searches have not found any nontrivial solutions.

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