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A note on certain inequalities for bivariate means

József Sándor

Department of Mathematics, Babeş-Bolyai University
Str. Kogalniceanu nr. 1, 400084 Cluj-Napoca, Romania
e-mail: jjsandor@hotmail.com

Abstract: We obtain simple proofs of certain results from paper [1].
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1 Introduction

Let a, b be two distinct positive numbers. The power mean of order k of a and b is defined by $A_k = A_k(a, b) = \left(\frac{a^k + b^k}{2}\right)^{1/k}$, $k \neq 0$ and $A_0 = \lim_{k\to 0} A_k = \sqrt{ab} = G(a, b)$. Let $A_1 = A$ denote also the classical arithmetic mean of a and b, and $He = He(a, b) = \frac{2A + G}{3} = \frac{a + b + \sqrt{ab}}{3}$ the so-called Heronian mean.

In the recent paper [1] the following results have been proved:

$$A_k(a, b) > a^{1-k} I(a^k, b^k) \text{ for } 0 < k \leq 1;\ b > a \tag{1}$$

$$A_k(a, b) < I(a, b) \text{ for } 0 < k \leq \frac{1}{2}; \tag{2}$$

$$He(a^k, b^k) < A_\beta(a^k, b^k) < \frac{3}{2^{1/\beta}} He(a^k, b^k) \text{ for } k > 0,\ \beta \geq \frac{2}{3} \tag{3}$$

and

$$A_k < S < 2^{1/k} \cdot A_k \text{ for } 1 \leq k \leq 2. \tag{4}$$

In the proofs of (1)–(4) the differential calculus has been used. Our aim will be to show that, relations (1)–(4) are easy consequences of some known results.

2 Main results

Lemma 2.1. *The function* $f_1(k) = \left(\frac{a^k+b^k}{2}\right)^{1/k} = A_k(a,b)$ *is a strictly increasing function of* k*; while* $f_2(k) = (a^k+b^k)^{1/k}$ *is a strictly decreasing function of* k*. Here* k *runs through the set of real numbers.*

Proof. Through these results are essentially known in the mathematical folklore, we shall give here a proof.

Simple computations yield:

$$k^2\frac{f_1'(k)}{f_1(k)} = \frac{x\ln x + y\ln y}{x+y} - \ln\left(\frac{x+y}{2}\right), \tag{5}$$

and

$$k^2\frac{f_2'(k)}{f_2(k)} = \frac{x\ln x + y\ln y}{x+y} - \ln(x+y), \tag{6}$$

where $x = a^k > 0$, $y = b^k > 0$. Since the function $f(x) = x\ln x$ is strictly convex (indeed: $f''(x) = \frac{1}{x} > 0$) by $f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2}$, relation (5) implies $f_1'(k) > 0$. Since the function $t \to \ln t$ is strictly increasing, one has $\ln x < \ln(x+y)$ and $\ln y < \ln(x+y)$; so $x\ln x + y\ln y < (x+y)\ln(x+y)$, so relation (6) implies that $f_2'(t) > 0$. These prove the stated monotonicity properties.

Proof of (1.1) By the known inequality $I < A$ we have $I(a^k,b^k) < A(a^k,b^k) = \frac{a^k+b^k}{2}$. Now $\frac{a^k+b^k}{2} \leq a^{k-1}\left(\frac{a^k+b^k}{2}\right)^{1/k}$ is equivalent with (for $1-k > 0$) $\left(\frac{a^k+b^k}{2}\right)^{1/k} > a$ or $a^k + b^k > 2a^k$, which is true by $b > a$. For $k = 1$ the inequality becomes $I < A$.

Proof of (1.2) Since A_k is strictly increasing, one has

$$A_k \leq A_{1/2} = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 = \frac{A+G}{2} < I,$$

by a known result (see [3]) of the author:

$$I > \frac{2A+G}{3} > \frac{A+G}{2}. \tag{7}$$

Proof of (1.3) By the inequality $He < A_{2/3}$ (see [2]) one has

$$He(a^k,b^k) < A_{2/3}(a^k,b^k) \leq A_\beta(a^k,b^k),$$

by the first part of Lemma 2.1.

Now,

$$2^{1/\beta}A_\beta(a^k,b^k) \leq 2^{3/2}(a^k,b^k)$$

by the second part of Lemma 2.1, and

$$A_{2/3}(a^k,b^k) < \frac{3}{2\sqrt{2}}He(a^k,b^k),$$

by (see [2])

$$A_{2/3} < \frac{3}{2\sqrt{2}} He. \tag{8}$$

Since $2^{3/2} = 2\sqrt{2}$, inequality (3) follows.

Proof of (1.4) In [2] it was proved that

$$A_2 < S < \sqrt{2} A_2. \tag{9}$$

Now, by Lemma 2.1 one has, as $k \leq 2$ that $A_k \leq A_2 < S$ and $\sqrt{2} A_2 \leq 2^{1/k} A_k$. Thus, by (9), relation (4) follows. We note that condition $1 \leq k$ is not necessary. □

References

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