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A note on the greatest prime factor

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Abstract: Let $k \geq 2$ a fixed positive integer. Let $P(n)$ be the greatest prime factor of a positive integer $n \geq 2$. Let $F_k(n)$ be the number of $2 \leq s \leq n$ such that $P(s) > \frac{s}{k}$. We prove the following asymptotic formula

$$F_k(n) \sim C_k \frac{n}{\log n},$$

where C_k is a constant defined in this article.
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1 Notation and Preliminary results

Let $P(n)$ be the greatest prime factor of a positive integer $n \geq 2$. Note that if n is prime then $P(n) = n$. Therefore $2 \leq P(n) \leq n$ for all $n \geq 2$.

Let $k \geq 2$ a fixed positive integer. Let $F_k(n)$ be the number of $2 \leq s \leq n$ such that $P(s) > \frac{s}{k}$.

In this article we prove the following asymptotic formula

$$F_k(n) \sim C_k \frac{n}{\log n},$$

where C_k is a constant defined below.

Let $\beta_k(x)$ be the set of positive integers not exceeding x such that in their prime factorization appear some prime p pertaining to the interval $\left(\frac{x}{k}, x\right]$. That is, $\beta_k(x)$ is the set of positive integers not exceeding x such that the greatest prime factor of these positive integers pertain to the interval $\left(\frac{x}{k}, x\right]$.

The number of positive integers pertaining to the set $\beta_k(x)$ we denote $B_k(x)$. It is well-known [1] the following formula

$$B_k(x) = B_k \frac{x}{\log x} + o\left(\frac{x}{\log x}\right), \tag{1}$$

where the constant $B_k = 1/2 + 1/3 + \cdots + 1/k$.

Let $\alpha_k(x)$ be the set of positive integers not exceeding x such that in their prime factorization only appear primes p pertaining to the interval $\left[0, \frac{x}{k}\right]$. That is, $\alpha_k(x)$ is the set of positive integers not exceeding x such that the greatest prime factor of these positive integers pertain to the interval $\left[0, \frac{x}{k}\right]$. We assume that 1 pertains to the set $\alpha_k(x)$. These numbers are called smooth numbers. The number of positive integers pertaining to the set $\alpha_k(x)$ we denote $A_k(x)$.

Note that the sets $\beta_k(x)$ and $\alpha_k(x)$ are disjoints and $\beta_k(x) \cup \alpha_k(x) = A$, where A is the set of positive integers s such that $1 \leq s \leq \lfloor x \rfloor$. Consequently $A_k(x) + B_k(x) = \lfloor x \rfloor$ and hence we have (see (1))

$$A_k(x) = x - B_k \frac{x}{\log x} + o\left(\frac{x}{\log x}\right).$$

Let us consider a prime p such that $2 \leq p \leq n$. The set of multiples of p not exceeding n will be denoted $A(n,p)$. Therefore

$$A(n,p) = \left\{p.1, p.2, p.3, \ldots, p. \left\lfloor \frac{n}{p} \right\rfloor\right\} \tag{2}$$

Let $B_1(n,p)$ be the set of positive integers not exceeding n such that the prime p is their greatest prime factor. We denote $B_2(n,p)$ the number of elements in the set $B_1(n,p)$. Note that $B_1(n,p) \subset A(n,p)$. Then

$$\sum_{2 \leq p \leq n} B_2(n,p) = n - 1$$

$$A_k(n) = 1 + \sum_{2 \leq p \leq \frac{n}{k}} B_2(n,p)$$

$$B_k(n) = \sum_{\frac{n}{k} < p \leq n} B_2(n,p) \tag{3}$$

The set of elements $s \in A(n,p)$ such that $p > \frac{s}{k}$ we denote $C_1(n,p)$. The number of elements in the set $C_1(n,p)$ we denote $C_2(n,p)$. Clearly $C_1(n,p) \subset A(n,p)$.

Let $\pi(x)$ be the prime counting function. We have (prime number Theorem)

$$\pi(x) \sim \frac{x}{\log x}. \tag{4}$$

2 Main result

Theorem 2.1. *Let $k \geq 2$ a fixed positive integer. We have the following asymptotic formula*

$$F_k(n) \sim C_k \frac{n}{\log n}, \tag{5}$$

where the constant $C_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k-1}$.

Proof: We have

$$\begin{aligned} F_k(n) &= \sum_{2\le p\le n}\left(\sum_{s\in B_1(n,p)\cap C_1(n,p)} 1\right) = \sum_{2\le p\le \frac{n}{k}}\left(\sum_{s\in B_1(n,p)\cap C_1(n,p)} 1\right) \\ &+ \sum_{\frac{n}{k}<p\le n}\left(\sum_{s\in B_1(n,p)\cap C_1(n,p)} 1\right) \end{aligned} \tag{6}$$

Let us consider a prime p fixed such that $\frac{n}{k} < p \le n$.

If $s \in A(n,p)$ then we have $p > \frac{n}{k} \ge \frac{s}{k}$. That is, $p > \frac{s}{k}$. Therefore $C_1(n,p) = A(n,p)$. Consequently (see (3) and (1))

$$\begin{aligned} &\sum_{\frac{n}{k}<p\le n}\left(\sum_{s\in B_1(n,p)\cap C_1(n,p)} 1\right) = \sum_{\frac{n}{k}<p\le n}\left(\sum_{s\in B_1(n,p)\cap A(n,p)} 1\right) \\ &= \sum_{\frac{n}{k}<p\le n}\left(\sum_{s\in B_1(n,p)} 1\right) = \sum_{\frac{n}{k}<p\le n} B_2(n,p) = B_k(n) \\ &= B_k\frac{n}{\log n} + o\left(\frac{n}{\log n}\right) \end{aligned} \tag{7}$$

Let us consider a prime p fixed such that $2 \le p \le \frac{n}{k}$. Note that this inequality implies that

$$\left\lfloor \frac{n}{p} \right\rfloor \ge k \tag{8}$$

Now, let us consider the inequality (where h is a positive integer)

$$\frac{s}{k} = \frac{p.h}{k} < p$$

This inequality has the solutions

$$h = 1, 2, \ldots, k-1$$

Therefore (see (8))

$$C_1(n,p) = \{p.1, p.2, \ldots, p(k-1)\}$$

and

$$C_2(n,p) = k-1 \tag{9}$$

Suppose that $k+1 \le p \le \frac{n}{k}$. Then p is the greatest prime factor of the elements in the set $C_1(n,p)$. Consequently

$$B_1(n,p) \cap C_1(n,p) = C_1(n,p) \tag{10}$$

On the other hand, if $2 \le p \le k$ then the number of elements in $B_1(n,p) \cap C_1(n,p)$ is less than or equal to $k-1$.

Therefore, we have (see (10), (9) and (4))

$$\begin{aligned} & N + \sum_{k+1 \le p \le \frac{n}{k}} \left(\sum_{s \in B_1(n,p) \cap C_1(n,p)} 1 \right) = N + \sum_{k+1 \le p \le \frac{n}{k}} \left(\sum_{s \in C_1(n,p)} 1 \right) \\ = \ & N + \sum_{k+1 \le p \le \frac{n}{k}} C_2(n,p) = N + (k-1) \sum_{k+1 \le p \le \frac{n}{k}} 1 \\ = \ & N + (k-1) \left(\pi \left(\frac{n}{k} \right) - \pi(k) \right) \sim \frac{k-1}{k} \frac{n}{\log n} \end{aligned}$$

where $N = \sum_{2 \le p \le k} \left(\sum_{s \in B_1(n,p) \cap C_1(n,p)} 1 \right)$.

That is

$$\sum_{2 \le p \le \frac{n}{k}} \left(\sum_{s \in B_1(n,p) \cap C_1(n,p)} 1 \right) = \left(1 - \frac{1}{k} \right) \frac{n}{\log n} + o \left(\frac{n}{\log n} \right) \tag{11}$$

Equations (6), (7) and (11) give (5). The theorem is proved. □

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References

[1] Jakimczuk, R., A note on the primes in the prime factorization of an integer, *International Mathematical Forum*, Vol. 7, 2012, 2005–2012.