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An equation involving Dedekind's function

Cristinel Mortici[1,2]

[1] Valahia University of Târgovişte, Department of Mathematics
Bd. Unirii 18, 130082, Târgovişte, Romania
[2] Academy of Romanian Scientists
Splaiul Independenţei 54, 050094 Bucharest, Romania
e-mail: cristinel.mortici@hotmail.com

Abstract: In this note we solve the equation

$$\frac{1}{\psi\left(a^2\right)}+\frac{1}{\psi\left(b^2\right)}+\frac{1}{\psi\left(c^2\right)}=\frac{1}{\psi\left(ab\right)}+\frac{1}{\psi\left(bc\right)}+\frac{1}{\psi\left(ca\right)},$$

where ψ is Dedekind's function.
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1 Introduction and Results

If $n \geq 2$ is integer, and $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$ is its decomposition in prime factors, then Dedekind's function ψ is defined by the formula

$$\psi(n) = n\left(1+\frac{1}{p_1}\right)\left(1+\frac{1}{p_2}\right)\cdots\left(1+\frac{1}{p_k}\right),$$

while $\psi\left(1\right) = 1$.

We prove the following result

Theorem 1. *For every integers $k, a, b \geq 2$, the following inequality holds true:*

$$\psi(ab) \geq \sqrt[k]{\psi(a^k)\psi(b^k)}. \tag{1}$$

This inequality is strict if $k \geq 3$.

Equality $\psi(ab) = \sqrt{\psi(a^2)\psi(b^2)}$ holds if and only if a and b have the same prime factors.

As a consequence, we prove the next

Theorem 2 *If integers $a, b, c \geq 2$ satisfy*

$$\frac{1}{\psi\left(a^2\right)}+\frac{1}{\psi\left(b^2\right)}+\frac{1}{\psi\left(c^2\right)}=\frac{1}{\psi\left(ab\right)}+\frac{1}{\psi\left(bc\right)}+\frac{1}{\psi\left(ca\right)}, \tag{2}$$

then $a = b = c$.

Similar results involving Euler totient function and other arithmetic functions were established in [1].

2 Proofs

Proof of Theorem 1. Let us denote

$$\psi(a)=a\cdot\prod_{i\in I}\left(1+\frac{1}{p_i}\right)\cdot\prod_{j\in J}\left(1+\frac{1}{q_j}\right)$$

$$\psi(b)=b\cdot\prod_{i\in I}\left(1+\frac{1}{p_i}\right)\cdot\prod_{s\in S}\left(1+\frac{1}{r_s}\right)$$

where p_i divide both a and b, while $(q_j, b) = 1$ and $(r_s, a) = 1$. Cases I or J or S empty are accepted.

As

$$\psi(a^k)=a^k\cdot\prod\left(1+\frac{1}{p_i}\right)\cdot\prod\left(1+\frac{1}{q_j}\right)$$

$$\psi(b^k)=b^k\cdot\prod\left(1+\frac{1}{p_i}\right)\cdot\prod\left(1+\frac{1}{r_s}\right)$$

$$\psi(ab)=ab\cdot\prod\left(1+\frac{1}{p_i}\right)\cdot\prod\left(1+\frac{1}{q_j}\right)\cdot\prod\left(1+\frac{1}{r_s}\right),$$

we have

$$\psi(a^k)\psi(b^k)=a^kb^k\prod\left(1+\frac{1}{p_i}\right)^2\cdot\prod\left(1+\frac{1}{q_j}\right)\cdot\prod\left(1+\frac{1}{r_s}\right)\leq$$

$$\leq a^kb^k\prod\left(1+\frac{1}{p_i}\right)^k\cdot\prod\left(1+\frac{1}{q_j}\right)^k\cdot\prod\left(1+\frac{1}{r_s}\right)^k=\psi^k(ab).$$

This follows from

$$\prod\left(1+\frac{1}{p_i}\right)^2\leq\prod\left(1+\frac{1}{p_i}\right)^k \tag{3}$$

and

$$\prod\left(1+\frac{1}{q_j}\right)\leq\prod\left(1+\frac{1}{q_j}\right)^k \text{ and } \prod\left(1+\frac{1}{r_s}\right)\leq\prod\left(1+\frac{1}{r_s}\right)^k. \tag{4}$$

Equality in (3) appears when $k = 2$ or $I = \emptyset$, and equality in (4) appears when $J = \emptyset$ and $S = \emptyset$. □

Proof of Theorem 2. Using (1) with $k = 2$ and AM-GM means inequality, we get

$$\frac{1}{\psi\left(ab\right)}\leq\sqrt{\frac{1}{\psi\left(a^2\right)}\cdot\frac{1}{\psi\left(b^2\right)}}\leq\frac{1}{2}\left(\frac{1}{\psi\left(a^2\right)}+\frac{1}{\psi\left(b^2\right)}\right).$$

By adding analogue inequalities

$$\frac{1}{\psi(bc)} \leq \frac{1}{2}\left(\frac{1}{\psi(b^2)} + \frac{1}{\psi(c^2)}\right).$$

$$\frac{1}{\psi(ca)} \leq \frac{1}{2}\left(\frac{1}{\psi(c^2)} + \frac{1}{\psi(a^2)}\right),$$

we deduce

$$\frac{1}{\psi(ab)} + \frac{1}{\psi(bc)} + \frac{1}{\psi(ca)} \leq \frac{1}{\psi(a^2)} + \frac{1}{\psi(b^2)} + \frac{1}{\psi(c^2)}.$$

Equation (2) is equality case in previous inequality. As stated in Theorem 1, equality holds when a, b, c have the same prime factors and moreover, $\psi(a^2) = \psi(b^2) = \psi(c^2)$. This attracts $a = b = c$ and the assertion is proved. □

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References

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