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Euler–Euclid's type proof of the infinitude of primes involving Möbius function

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Dedicated to the advisor of my PhD thesis Prof. Dr. Žarko Pavićević on the occasion of his 60th birthday

Abstract: If we suppose that $S = \{p_1, p_2, \dots, p_k\}$ is a set of all primes, then taking $x = p_1p_2\cdots p_k + 1$ into a formula due to E. Meissel in 1854 gives

$$(p_1 - 1)(p_2 - 1)\cdots(p_k - 1) = 0.$$

This obvious contradiction yields the infinitude of primes.
Keywords: Euclid's theorem, infinitude of primes, Euclid's proof, Euler's proof(s), Möbius inversion formula, Meissel formula.
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1 Introduction

Ever since Euclid of Alexandria, sometimes before 300 B.C., first proved that the number of primes is infinite (see Proposition 20 in Book IX of his legendary *Elements* [6], mathematicians have amused themselves by coming up with alternate proofs. For more information about the Euclid's proof of the infinitude of primes see e.g., [2, page 414, Ch. XVIII] and [9, Section 1].

Euclid's proof of the infinitude of primes is a paragon of simplicity: *given a finite list of primes* $p_1, p_2, \dots p_k$*, multiply them together and add one. The resulting number, say* $N = p_1p_2\cdots p_k + 1$*, is not divisible by any prime on the list, so any prime factor of* N *is a new prime.* This yields

Euclid's theorem. *There are infinitely many primes.*

In [9] the author of this article provided a comprehensive historical survey of different proofs of famous Euclid's theorem on the infinitude of primes which has fascinated generations of mathematicians since its first and famous demonstration given by Euclid. A more sophisticated proof of Euclid's theorem was given after about two millennia ago by the Swiss mathematician Leonhard Euler. In 1737 Euler in his work [5, pp. 172–174] (also see [3]) showed that by adding the reciprocals of successive prime numbers you can attain a sum greater than any prescribed number; that is, in terms of modern Analysis, the sum of the reciprocals of all the primes diverges (cf. [10, page 11, Theorem 1.4] or [12, page 8]).

Another less known proof due to Euler in 1736 (published posthumously in 1862 [4]; also see [2, page 413]) is in fact the first proof of Euclid's theorem after those of Euclid's and C. Goldbach's proof presented in a letter to L. Euler in July 1730 (see [12, page 6] and [9, Appendix C)]). This proof is based on the *Euler's totient function* $\varphi(n)$, defined as the number of positive integers not exceeding n and relatively prime to n; for a proof see e.g., [1, pp. 134–135] or [11, page 3]. As noticed in Dickson's History [2, page 413] (see also [13, page 80]), this proof is also attributed in 1878/9 by Kummer [7] who gave essentially Euler's argument.

Motivated by the arguments used in Euclid's proof and the mentioned Euler's proof (that is, taking $x = p_1p_2\cdots p_k + 1$ into Meissel formula to obtain a contradiction $\varphi(x-1) = 0$), in the next section we present a new proof of Euclid's theorem.

2 Euler–Euclid's type proof of the infinitude of primes

Proof of Euclid's theorem. Recall that the *Möbius μ-function* from Elementary Number Theory is defined so that $\mu(1) = 1$, $\mu(n) = (-1)^k$ if n is a product of k distinct primes, and $\mu(n) = 0$ if n is divisible by the square of a prime. Applying the well known *Möbius inversion formula* (see e.g., [14, page 62, Theorem 3.5.1]) to the the *greatest integer function* $f(x) := \sum_{n\le x} 1 = [x]$ $(1 \le x < +\infty)$, where the sum ranges over all positive integers n that are $\le x$, we immediately obtain the following formula established in 1854 by E. Meissel [8] (cf. also [14, page 65, the formula (3.5.14)]):

$$\sum_{n\le x}\mu(n)\left[\frac{x}{n}\right] = 1 \quad (1 \le x < +\infty). \tag{1}$$

Suppose that $S = \{p_1, p_2, \ldots, p_k\}$ is a set of all the primes. Then since $\mu(n) \ne 0$ if and only if $n = 1$ or n is a product of certain (distinct) elements of S, taking $x = p_1p_2\cdots p_k + 1$ into (1), it easily reduces to

$$(-1)^k + \sum_{s=1}^{k}(-1)^{k-s}\sum_{1\le i_1<i_2<\cdots<i_s\le k} p_{i_1}p_{i_2}\cdots p_{i_s} = 0,$$

which clearly can be written as

$$(p_1-1)(p_2-1)\cdots(p_k-1) = 0.$$

This obvious contradiction yields the infinitude of primes. □

Remark. Recall that the *prime-counting function* $\pi(x)$ is defined as the number of primes not exceeding x (x is any real number). If for given positive integer $n \geq 4$, Δ denotes the product of all primes less than or equal $\sqrt{n}$, then by the *Legendre's formula* stated in the modern form,

$$\pi(n) - \pi(\sqrt{n}) = \sum_{d|\Delta} \mu(d) \left[\frac{x}{d}\right] - 1 \quad (1 \leq x < +\infty). \tag{2}$$

The classical proof of this formula uses the *inclusion-exclusion principle* (see e.g., [10, pp. 33–34, Theorem 1.17]). If $p_1 < p_2 < \cdots < p_k$ are all the primes, then since $\pi(p_k^2) = \pi(p_k) = k$, taking $n = p_k^2$ into (2), it immediately reduces to the Meissel formula (1).

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