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Sieving $2m$-prime pairs

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Abstract: A new characterization of $2m$-prime pairs is obtained. In particular, twin prime pairs are characterized. Our results give elementary methods for finding $2m$-prime pairs (e.g. twin prime pairs) up to a given integer.
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1 Twin prime pairs

A *twin prime pair* is a pair of prime numbers of the form $(p, p+2)$. Except 2 and 3 each prime number is of the form $6k-1$ or $6k+1$ and hence each twin prime pair, except (3,5), is of the form $(6k-1, 6k+1)$ for some positive integer k. In this section we shall present an elementary method for finding twin prime pairs up to a given integer. Our method is based on the following results. Their proofs will be given in the second section, where more general results on $2m$-prime pairs are obtained.

Lemma 1.1. *Let p be a prime. If $p = 6j+1$ or $p = 6j-1$ then for each $i \geq 1$ both*

$$(6(pi+j)-1, 6(pi+j)+1) \text{ and } (6(pi-j)-1, 6(pi-j)+1)$$

are not twin prime pairs.

Theorem 1.2. *Let k be a positive integer. Then $(6k-1, 6k+1)$ is not a twin prime pair if and only if there exist a prime $p \leqslant \sqrt{6k+1}$ of the form $6j \pm 1$ and a positive integer i such that either $k = pi + j$ or $k = pi - j$.*

Let us now describe our method for sieving twin prime pairs up to a given positive integer n.

1. `Write down a list of all integers` $k = 1, 2, \ldots, \lceil \frac{n}{6} \rceil$.

2. `Find all primes` $3 < p \leqslant \sqrt{n}$.
   For example, one can apply the Sieve of Erathostenes. Note that it suffices to proceed as long as the primes are not greater then $\sqrt{n}$, because in the prime factorization of n at least one factor is not greater then $\sqrt{n}$ (just like in the case when using the Sieve of Erathostenes in order find primes up to n).

3. `For each prime` $3 < p \leqslant \sqrt{n}$ `we do the following:`

   - `if` $6 \mid p+1$ `then` $j = \frac{p+1}{6}$, `else` $j = \frac{p-1}{6}$;
   - `cross out integers` $k = pi + j$ `and` $k = pi - j$ `for all` $i = 1, 2, ...,$ `from our list.`

4. `Each remaining integer` k `in the list gives us a twin prime pair` $(6k-1, 6k+1)$.

According to Theorem 1.2 this method returns all twin prime pairs up to n, except (3,5). In the following example we apply our method to the case when $n = 250$.

Example 1.3. *Let us find all twin prime pairs up to* 250*. We list all integers* $k = 1, 2, \ldots, 42$*. Next, we find all primes* $3 < p \leq \sqrt{250}$*, these are,* $5, 7, 11, 13$*.*

(i) For $p = 5 = 6 \cdot 1 - 1$ *we have* $j = 1$ *and hence we cross out all integers* k *of the form* $5i - 1$ *and* $5i + 1$ *from our list.*

(ii) For $p = 7 = 6 \cdot 1 + 1$ *we have* $j = 1$ *and hence we cross out all integers* k *of the form* $7i - 1$ *and* $7i + 1$ *from our list.*

(iii) For $p = 11 = 6 \cdot 2 - 1$ *we have* $j = 2$ *and hence we cross out all integers* k *of the form* $11i - 2$ *and* $11i + 2$ *from our list.*

(iv) For $p = 13 = 6 \cdot 2 + 1$ *we have* $j = 2$ *and hence we cross out all integers* k *of the form* $13i - 2$ *and* $13i + 2$ *from our list.*

1	*2*	*3*	*4*	*5*	~~*6*~~	*7*	~~*8*~~	~~*9*~~	*10*
~~*11*~~	*12*	~~*13*~~	~~*14*~~	~~*15*~~	~~*16*~~	*17*	*18*	~~*19*~~	~~*20*~~
~~*21*~~	~~*22*~~	*23*	~~*24*~~	*25*	~~*26*~~	~~*27*~~	~~*28*~~	~~*29*~~	*30*
~~*31*~~	*32*	*33*	~~*34*~~	~~*35*~~	~~*36*~~	~~*37*~~	*38*	~~*39*~~	*40*
~~*41*~~	~~*42*~~								

For each remaining integer k *in our list we get a twin prime pair* $(6k-1, 6k+1)$*. Thus, adding* $(3,5)$*, we obtain all twin prime pairs up to* 250*:*

$(3,5)$, $(5,7)$, $(11,13)$, $(17,19)$, $(29,31)$, $(41,43)$, $(59,61)$, $(71,73)$, $(101,103)$, $(107,109)$, $(137,139)$, $(149,151)$, $(179,181)$, $(191,193)$, $(197,199)$, $(227,229)$, $(239,241)$.

There are also some other algorithms for searching twin prime pairs. However, in most cases they are not elementary. Among those that are elementary one usually has to find all primes up to n, in order to find all twin prime pairs up to n. Note that in our algorithm presented above we only have to find all primes up to $\sqrt{n}$, in order to find all twin prime pairs up to n. This could be a substantial advantage when dealing with larger values of n.

2 $2m$-prime pairs

Let m be a positive integer. A *$2m$-prime pair* is a pair of primes of the form $(p, p+2m)$. For $m = 1$ we get 2-prime pairs, which are ordinary twin prime pairs: $(3, 5)$, $(5, 7)$, $(11, 13)$, $(17, 19)$, $(29, 31)$, $(41, 43)$, ... For $m = 2$ we get 4-prime pairs (also known as cousin prime pairs): $(3, 7)$, $(7, 11)$, $(13, 17)$, $(19, 23)$, ... For $m = 3$ we get 6-prime pairs (also known as sexy prime pairs): $(5, 11)$, $(7, 13)$, $(11, 17)$, $(13, 19)$, ... For $m = 4$ we get 8-prime pairs: $(3, 11)$, $(5, 13)$, $(11, 19)$, $(23, 31)$, ... etc.

Obviously, we can consider twin prime pairs as a special case of $2m$-prime pairs. Thus, the well known conjecture that there are infinitely many twin prime pairs can be generalized to all $2m$-prime pairs. Namely, we conjecture that each even positive integer can be written as the difference of two primes in infinitely many ways. This conjecture is closely related to a more stronger de Polignac's [1] conjecture from 1849 that each even positive integer can be written as the difference of two consecutive primes in infinitely many ways. Unfortunately, de Polinac's conjecture has not been settled yet. However, in 2013 Zhang [2] proved that there exists an even positive integer $n < 7 \cdot 10^7$, which can be written as the difference of two consecutive primes in infinitely many ways.

In this section we show that we are able to obtain sieves for sieving $2m$-prime pairs, where m is an arbitrary positive integer. This can be done in a similar manner as the twin prime sieve was obtained in the previous section. Before doing that we have to divide all $2m$-prime pairs into the following four groups (here we ignore the prime 3, which is an exception since it is congruent neither to -1 nor to 1 modulo 6):

(i) $(6n - 4)$-prime pairs (e.g. 2-prime pairs, 8-prime pairs, ...).
The first member of each such pair is always congruent to -1 and the second one to 1 modulo 6. Thus, each such prime pair is of the form: $(6k - 1, 6k + 6n - 5)$ for some positive integers n and k.

(ii) $(6n - 2)$-prime pairs (e.g. 4-prime pairs, 10-prime pairs, ...).
The first member of each such pair is always congruent to 1 and the second one to -1 modulo 6. Thus, each such prime pair is of the form: $(6k + 1, 6k + 6n - 1)$ for some positive integers n and k.

(iii) $6n$-prime pairs, whose both members are congruent to -1 modulo 6. These are of the form: $(6k - 1, 6k + 6n - 1)$ for some positive integers n and k.

(iv) $6n$-prime pairs, whose both members are congruent to 1 modulo 6. These are of the form: $(6k+1, 6k+6n+1)$ for some positive integers n and k.

Our sieves for sieving $2m$-prime pairs can be obtained according to the following results. We start with the following generalization of Lemma 1.1.

Lemma 2.1. *Let n be a positive integer. If $p = 6j-1$ is a prime, then both*

$$\big(6\,(pi+j)-1, 6\,(pi+j)+6n-5\big)$$

and

$$\big(6\,(pi-j-n+1)-1, 6\,(pi-j-n+1)+6n-5\big)$$

are not $(6n-4)$-prime pairs for each $i \geq 1$. If $p = 6j+1$ is a prime, then both

$$\big(6\,(pi-j)-1, 6\,(pi-j)+6n-5\big)$$

and

$$\big(6\,(pi+j-n+1)-1, 6\,(pi+j-n+1)+6n-5\big)$$

are not $(6n-4)$-prime pairs for each $i \geq 1$.

Proof. First, suppose that $p = 6j-1$. Then $6\,(pi+j)-1 = 6pi+p = p\,(6i+1)$ and $6\,(pi-j-n+1)+6n-5 = 6pi-p = p\,(6i-1)$ are composite for each $i \geq 1$. Similarly, if $p = 6j+1$ we see that $6\,(pi-j)-1$ and $6\,(pi+j-n+1)+6n-5$ are composite for each $i \geq 1$. $\square$

Lemma 2.2. *Let k be a positive integer. If $6k-1$ is not a prime, then there exist positive integers i and j such that one of the following holds true:*

(i) $p := 6j-1$ *is a prime and* $k = pi+j$*,*

(ii) $p := 6j+1$ *is a prime and* $k = pi-j$*.*

In both cases $p \leqslant \sqrt{6k-1}$ (hence, $p \leqslant \sqrt{6k+6n-5}$ for each positive integer n).

Proof. Suppose that $6k-1$ is not prime. Then there is a prime $p \leqslant \sqrt{6k-1}$ such that $p \mid 6k-1$. Thus, $p \neq 2, 3$. Using the division algorithm there exist nonnegative integers n, r such that $k = pn+r$ and $0 \leqslant r < p$. Consequently, $p \mid 6\,(pn+r)-1 = 6pn+(6r-1)$ and hence $p \mid 6r-1$. Since $6r-1 = pt$ for some positive integer t and $r < p$ we see that

$$t = \frac{6r-1}{p} < \frac{6p-1}{p} < 6.$$

Thus, $t \in \{1,2,3,4,5\}$ and $6r-tp = 1$. This means that 6 and t are relatively prime and so $t = 1$ or $t = 5$. Since $p \neq 2, 3$ we have either $p = 6j-1$ or $p = 6j+1$ for some j.
We first consider the case when $p = 6j-1$. If $t = 5$ we would have $1 = 6r-5p \equiv p \,(\mathrm{mod}\, 6)$, a contradiction. Thus, $t = 1$ and hence $p = 6r-1$. Therefore, $r = j$ and so $k = pi+j$ for $i := n$.

Moreover, $i > 0$. Namely, if $i = 0$ we would have $k = r$ and so $6k - 1 = 6r - 1 = p$, which is impossible. Thus, (i) holds true.

Next, let $p = 6j + 1$. If $t = 1$ we would have $p = 6r - 1 \equiv -1 \pmod 6$, a contradiction. Thus, $t = 5$ and hence $6r - 1 = 5p = 30j + 5$. Therefore, $r = 5j + 1 = p - j$ and so $k = pn + p - j = pi - j$ for $i := n + 1 > 0$. Thus, (ii) holds true. □

Lemma 2.3. *Let k and n be positive integers. If $6k + 6n - 5$ is not a prime, then there exist positive integers i and j such that one of the following holds true:*

(i) $p := 6j - 1$ *is a prime and* $k = pi - j - n + 1$*,*

(ii) $p := 6j + 1$ *is a prime and* $k = pi + j - n + 1$*.*

In both cases $p \leqslant \sqrt{6k + 6n - 5}$*.*

Proof. Suppose that $6k + 6n - 5$ is not a prime. Then there is a prime $p \leqslant \sqrt{6k + 6n - 5}$, such that $p \mid 6k + 6n - 5$. Thus $p \neq 2, 3$. Using the division algorithm there exist nonnegative integers m, r, such that $k = pm + r$ and $0 \leqslant r < p$. Consequently, $p \mid 6(pm + r) + 6n - 5 = 6pm + (6r + 6n - 5)$ and hence $p \mid 6r + 6n - 5$. Thus, $6r + 6n - 5 = pt$ for some positive integer t. Since $6r + 6n - 5$ is not divisible by 2 and 3, we see that t and 6 are relatively prime. Hence, $t \equiv -1 \pmod 6$ or $t \equiv 1 \pmod 6$. Since $p \neq 2, 3$ it follows that $p = 6j - 1$ or $p = 6j + 1$ for some positive integer j.

We first consider the case when $p = 6j - 1$. Suppose that $t \equiv 1 \pmod 6$. Then $6r + 6n - 5 = (6j - 1)(6u + 1)$ for some nonnegative integer u, which yields $1 \equiv -1 \pmod 6$, a contradiction. Hence, $t \equiv -1 \pmod 6$. Let u be a positive integer such that $t = 6u - 1$. Then $6r + 6n - 5 = (6j - 1)(6u - 1) = 36ju - 6j - 6u + 1$ and so

$$r = 6ju - u - j - n + 1 = up - j - n + 1.$$

Thus,

$$k = pm + r = pm + up - j - n + 1 = p(m + u) - j - n + 1 = pi - j - n + 1$$

for $i := m + u$, which is a positive integer. So we see that (i) holds true.

Next, let us assume that $p = 6j + 1$. If $t \equiv -1 \pmod 6$, then $6r + 6n - 5 = (6j + 1)(6u - 1)$ for some positive integer u, which implies that $1 \equiv -1 \pmod 6$, a contradiction. Therefore, $t \equiv 1 \pmod 6$. Then $6r + 6n - 5 = (6j + 1)(6u + 1) = 36ju + 6j + 6u + 1$ for some nonnegative integer u. Consequently,

$$r = 6ju + u + j - n + 1 = up + j - n + 1.$$

and so

$$k = pm + r = pm + up + j - n + 1 = p(m + u) + j - n + 1 = pi + j - n + 1$$

for $i := m + u$, which is a nonnegative integer. Suppose that $i = 0$. Then $m = u = 0$, and hence $k = r$ and $r = j - n + 1$. This means that

$$6k + 6n - 5 = 6r + 6n - 5 = 6(j - n + 1) + 6n - 5 = 6j + 1 = p$$

is a prime, a contradiction. Therefore, i is a positive integer. Thus, we see that (ii) holds true. □

Theorem 2.4. *Let k and n be positive integers. Then $(6k-1, 6k+6n-5)$ is not a $(6n-4)$-prime pair if and only if there exist positive integers i and j such that one of the following holds true:*

(i) $p := 6j-1$ *is a prime and* $k = pi + j$ *or* $k = pi - j - n + 1$,

(ii) $p := 6j+1$ *is a prime and* $k = pi - j$ *or* $k = pi + j - n + 1$.

In both cases $p \leqslant \sqrt{6k+6n-5}$.

Proof. Note that one implication follows from Lemma 2.1 and the other one from Lemma 2.2 and Lemma 2.3. □

Remark 2.5. Obviously, if $n = 1$ we get Theorem 1.2. Thus, Theorem 2.4 is a generalization of Theorem 1.2.

Similarly as in the case of $(6n-4)$-prime pairs one can obtain the following theorems for all other types of $2m$-prime pairs. We omit their proofs since they are all very similar to the proof of Theorem 2.4.

Theorem 2.6. *Let k and n be positive integers. Then $(6k+1, 6k+6n-1)$ is not a $(6n-2)$-prime pair if and only if there exist positive integers i and j such that one of the following holds true:*

(i) $p := 6j-1$ *is a prime and* $k = pi - j$ *or* $k = pi + j - n$,

(ii) $p := 6j+1$ *is a prime and* $k = pi + j$ *or* $k = pi - j - n$.

In both cases $p \leqslant \sqrt{6k+6n-1}$.

Theorem 2.7. *Let k and n be positive integers. Then $(6k-1, 6k+6n-1)$ is not a $(6n)$-prime pair if and only if there exist positive integers i and j such that one of the following holds true:*

(i) $p := 6j-1$ *is a prime and* $k = pi + j$ *or* $k = pi + j - n$,

(ii) $p := 6j+1$ *is a prime and* $k = pi - j$ *or* $k = pi - j - n$.

In both cases $p \leqslant \sqrt{6k+6n-1}$.

Theorem 2.8. *Let k and n be positive integers. Then $(6k+1, 6k+6n+1)$ is not a $(6n)$-prime pair if and only if there exist positive integers i and j such that one of the following holds true:*

(i) $p := 6j-1$ *is a prime and* $k = pi - j$ *or* $k = pi - j - n$,

(ii) $p := 6j+1$ *is a prime and* $k = pi + j$ *or* $k = pi + j - n$.

In both cases $p \leqslant \sqrt{6k+6n+1}$.

Remark 2.9. For each positive integer m these four theorems allow us to create a sieve for sieving $2m$-prime pairs. These sieves can be obtained in an analogous manner as our algorithm for sieving twin prime pairs in the first section of this paper. Note that all these sieves are not able to detect $2m$-prime pairs of the form $(3, 3+2m)$. Namely, 3 is the only odd prime which is congruent neither to -1 nor to 1 modulo 6, and so we have to check extra if $(3, 3+2m)$ is a $2m$-prime pair or not.

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