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Note on φ, ψ and σ-functions. Part 7

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Abstract: The inequality

$$\frac{\Omega(n) - \omega(n)}{2^{\Omega(n)-\omega(n)}}\varphi(n) \leq \sigma(n) - \psi(n) < 2^{\Omega(n)-1}\varphi(n)$$

connecting φ, ψ and σ-functions is formulated and proved.
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1 Main results

Let us have the natural number $n \geq 2$ with canonical representation

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where k, α_1, ... $\alpha_k \geq 1$ are natural numbers and $p_1, ..., p_k$ are different prime numbers. Let us define for n the following functions (cf., e.g, [1, 2]):

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i - 1),\ \varphi(1) = 1,$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i + 1),\ \psi(1) = 1,$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1},\ \sigma(1) = 1,$$

$$\Omega(n) = \sum_{i=1}^{k} \alpha_i,$$

$$\omega(n) = k,$$

$$\underline{set}(n) = \{p_1, ..., p_k\}.$$

Theorem: For every natural number $n \geq 2$

$$\frac{\Omega(n) - \omega(n)}{2^{\Omega(n)-\omega(n)}}\varphi(n) \leq \sigma(n) - \psi(n) < 2^{\Omega(n)-1}\varphi(n). \tag{1}$$

Proof: Let the natural number n be a prime. Then $\Omega(n) - \omega(n) = 0$ and

$$\frac{\Omega(n) - \omega(n)}{2^{\Omega(n)-\omega(n)}}\varphi(n) = 0 = \sigma(n) - \psi(n) < 2^{\Omega(n)-1}\varphi(n),$$

i.e., (1) holds.

Let us prove the left inequality in (1), i.e.

$$\frac{\Omega(n) - \omega(n)}{2^{\Omega(n)-\omega(n)}}\varphi(n) \leq \sigma(n) - \psi(n). \tag{2}$$

Above, we mentioned the case when $\Omega(n) = 1$ is valid. Let us assume that (2) is valid for every natural number n with $\Omega(n) = m$ for some natural number $m \geq 2$. Let p be a prime number. Then $\Omega(np) = \Omega(n) + 1$.

For p there are two cases. In the first case, $p \notin \underline{set}(n)$. Then $\omega(np) = \omega(n) + 1$ and

$$\sigma(np) - \psi(np) - \frac{\Omega(np) - \omega(np)}{2^{\Omega(np)-\omega(np)}}\varphi(np)$$

$$= (\sigma(n) - \psi(n)).(p + 1) - \frac{\Omega(n) - \omega(n)}{2^{\Omega(n)-\omega(n)}}\varphi(n).(p - 1)$$

(from (2))

$$\geq \frac{\Omega(n) - \omega(n)}{2^{\Omega(n)-\omega(n)}}\varphi(n).(p + 1) - \frac{\Omega(n) - \omega(n)}{2^{\Omega(n)-\omega(n)}}\varphi(n).(p - 1)$$

$$= 2\frac{\Omega(n) - \omega(n)}{2^{\Omega(n)-\omega(n)}}\varphi(n) \geq 0.$$

In the second case, $p \in \underline{set}(n)$. Then $\omega(np) = \omega(n)$. From $\sigma(np) > p\sigma(n)$ we obtain

$$\sigma(np) - \psi(np) - \frac{\Omega(np) - \omega(np)}{2^{\Omega(np)-\omega(np)}}\varphi(np)$$

$$> p.(\sigma(n) - \psi(n) - \frac{\Omega(n) - \omega(n) + 1}{2^{\Omega(np)-\omega(np)+1}}\varphi(n))$$

$$\geq p.(\frac{\Omega(n) - \omega(n)}{2^{\Omega(n)-\omega(n)}} - \frac{\Omega(n) - \omega(n) + 1}{2^{\Omega(np)-\omega(np)+1}}).\varphi(n)) \geq 0,$$

because for each natural number $a \geq 1$

$$\frac{a}{2^a} - \frac{a + 1}{2^{a+1}} = \frac{a - 1}{2^{a+1}} \geq 0.$$

Therefore (2) is proved. Now we prove the right inequality in (1), i.e.

$$\sigma(n) - \psi(n) < 2^{\Omega(n)-1}\varphi(n). \tag{3}$$

First, we see that for $n = 2^a$, where the natural number $a \geq 2$:

$$2^{\Omega(2^a)-1}\varphi(2^a) - \sigma(2^a) + \psi(2^a)$$

$$= 2^{a-1}.2^{a-1} - 2^{a+1} + 1 + 3.2^{a-1} > 2.2^{a-1} - 4.2^{a-1} + 1 + 3.2^{a-1} > 0.$$

As we saw above, (3) is valid for $n = 2p$.

When $p \geq 3$ is a prime number and $a \geq 2$, then for $n = 2^a p$ we obtain:

$$2^{\Omega(2^a p)-1}\varphi(2^a p) - \sigma(2^a p) + \psi(2^a p)$$

$$= 2^a.2^{a-1}(p-1) - (2^{a+1}-1)(p+1) + 3.2^{a-1}(p+1)$$

(for $a \geq 2$)

$$\geq 4.2^{a-1}(p-1) - 2^{a-1}(p+1) > 0.$$

Let us assume that (3) is valid for every odd number n with $\Omega(n) = m$ for some natural number $m \geq a+1$, where $2^a | n$ and $2^{a+1} \not| n$ for the natural number $a \geq 2$. Let $p \geq 3$ be a prime number. Then, as above, $\Omega(np) = \Omega(n) + 1$.

For $p \geq 3$, again, there are two cases. In the first case, $p \notin \underline{set}(n)$. Then

$$2^{\Omega(np)-1}\varphi(np) - \sigma(np) + \psi(np)$$

$$= 2^{\Omega(n)}\varphi(n).(p-1) - \sigma(n)(p+1) + \psi(n)(p+1)$$

(from (3) we have that $\sigma(n) < 2^{\Omega(n)-1}\varphi(n) + \psi(n)$)

$$\geq 2^{\Omega(n)}\varphi(n).(p-1) - 2^{\Omega(n)-1}\varphi(n)(p+1) - \psi(n)(p+1) + \psi(n)(p+1)$$

$$> 2^{\Omega(n)-1}\varphi(n).(2(p-1) - (p+1))$$

$$= 2^{\Omega(n)-1}\varphi(n)(p-3) \geq 0.$$

In the second case, $p \in \underline{set}(n)$. Then $n = 2^a p^b.s$ for the above a and for some natural numbers $b \geq 1$ and $s \geq 1$.

Let $y = 2^a s$. Then

$$\sigma(y) = \sigma(2^a s) = \frac{2^{a+1}-1}{3.2^{a-1}}.3.2^{a-1}\sigma(s) \geq \frac{2^{a+1}-1}{3.2^{a-1}}.3.2^{a-1}\psi(s)$$

$$= \frac{2^{a+1}-1}{3.2^{a-1}}\psi(2^a s) = \frac{2^{a+1}-1}{3.2^{a-1}}\psi(y). \tag{4}$$

Now,

$$2^{\Omega(np)-1}\varphi(np) - \sigma(np) + \psi(np)$$

$$= 2^{\Omega(n)}\varphi(n).p - \sigma(y).\frac{p^{b+2}-1}{p-1} + \psi(n).p$$

$$> 2(\sigma(n) - \psi(n)).p - \sigma(y).\frac{p^{b+2}-1}{p-1} + p^b(p+1)\psi(y)$$

$$= 2(\sigma(y).\frac{p^{b+1}-1}{p-1}.p - p^b(p+1)\psi(y)) - \sigma(y).\frac{p^{b+2}-1}{p-1} + p^b(p+1)\psi(y)$$

$$= (2\frac{p^{b+1}-1}{p-1}.p - \frac{p^{b+2}-1}{p-1})\sigma(y) - p^b(p+1)\psi(y)$$

$$= \frac{p^{b+2}-2p+1}{p-1}.\sigma(y) - p^b(p+1)\psi(y)$$

(from (4))

$$\geq \frac{p^{b+2}-2p+1}{p-1}.\frac{2^{a+1}-1}{3.2^{a-1}}\psi(y) - p^b(p+1)\psi(y)$$

$$= \frac{\psi(y)}{3.2^{a-1}(p-1)}(2^{a+1}p^{b+2} - 2^{a+2}p + 2^{a+1} - p^{b+2} + 2p - 1 - 3.2^{a-1}p^{b+2} + 3.2^{a-1}p^b)$$

$$> \frac{\psi(y)}{3.2^{a-1}(p-1)}((2^{a-1}-1)p^{b+2} - 2^{a+2}p + 2p + 3.2^{a-1}p^b)$$

(from $b \geq 1$)

$$> \frac{\psi(y)}{3.2^{a-1}(p-1)}((2^{a-1}-1)p^3 - 8.2^{a-1}p + 2p + 3.2^{a-1}p)$$

(from $p \geq 3$)

$$> \frac{\psi(y)}{3.2^{a-1}(p-1)}(9(2^{a-1}-1)p - 5.2^{a-1}p + 2p)$$

$$= \frac{\psi(y)}{3.2^{a-1}(p-1)}(4.2^{a-1}p - 7p)$$

(from $a \geq 2$)

$$= \frac{\psi(y)}{3.2^{a-1}(p-1)}(8p - 7p) > 0.$$

Therefore, (3) and hence (1), are proved. □

References

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[2] Nagell, T. *Introduction to Number Theory*, John Wiley & Sons, New York, 1950.