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A note on the number of perfect powers in short intervals

Rafael Jakimczuk

División Matemática, Universidad Nacional de Luján
Buenos Aires, Argentina
e-mail: jakimczu@mail.unlu.edu.ar

Abstract: Let $N(x)$ be the number of perfect powers that do not exceed x. In this note we obtain asymptotic formulae for the difference $N(x + x^{\theta}) - N(x)$, where $1/2 < \theta < 2/3 + 1/7$. We also prove that if $\theta = 1/2$ the difference $N(x + x^{\theta}) - N(x)$ is zero for infinite x arbitrarily large.
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1 Preliminary results

A natural number of the form m^n where m is a positive integer and $n \geq 2$ is called a perfect power. The first few terms of the integer sequence of perfect powers are

$$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128 \ldots$$

In this article, $N(x)$ denotes the number of perfect powers that do not exceed x. That is, the perfect power counting function.

Let p_n be the n-th prime. Consequently we have,

$$p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, p_6 = 13, \ldots$$

Jakimczuk [1] proved the following theorem.

Theorem 1.1. *Let p_n $(n \geq 2)$ be the n-th prime. The following asymptotic formula holds*

$$N(x) = \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1,\ p_{i_1} \cdots p_{i_k} < p_n} x^{\frac{1}{p_{i_1} \cdots p_{i_k}}} + g(x) x^{\frac{1}{p_n}}, \tag{1}$$

where $\lim_{x\to\infty} g(x) = 1$. *The expression* $1 \le i_1 < \cdots < i_k \le n-1,\ p_{i_1}\dots p_{i_k} < p_n$ *indicates that the sum is taken over the k-element subsets* $\{i_1,\dots,i_k\}$ *of the set* $\{1,2,\dots,n-1\}$ *such that the inequality* $p_{i_1}\dots p_{i_k} < p_n$ *holds.*

For example:

If $n = 4$ then Theorem 1.1 becomes,

$$N(x) = \sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x} - \sqrt[6]{x} + g(x)\sqrt[7]{x},$$

where $\lim_{x\to\infty} g(x) = 1$.

If $n = 5$ then Theorem 1.1 becomes,

$$N(x) = \sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x} - \sqrt[6]{x} + \sqrt[7]{x} - \sqrt[10]{x} + g(x)\sqrt[11]{x},$$

where $\lim_{x\to\infty} g(x) = 1$.

The following lemma is an immediate consequence of the binomial Theorem.

Lemma 1.2. *We have the following formula*

$$(1+x)^{1/\alpha} = 1 + \frac{1}{\alpha}x + f_\alpha(x)x^2,$$

where

$$\lim_{x\to 0} f_\alpha(x) = \frac{1}{2}\frac{1}{\alpha}\left(\frac{1}{\alpha} - 1\right)$$

2 Main results

Lemma 2.1. *If* $n \ge 4$ *(that is* $p_n \ge 7$*) and* $0 < \lambda < \frac{1}{6}$ *are fixed numbers we have the following asymptotic formula,*

$$N(x + x^{\frac{1}{2}+\frac{1}{p_n}+\lambda}) = N(x) + \frac{1}{2}x^{\frac{1}{p_n}+\lambda} + o\left(x^{\frac{1}{p_n}}\right). \tag{2}$$

Proof. Let us consider the function $g(x)$ (see (1)). Suppose that $0 < \beta < 1$. We have (Lemma 1.2)

$$\begin{aligned}
& g(x+x^\beta)(x+x^\beta)^{\frac{1}{p_n}} = g(x+x^\beta)x^{\frac{1}{p_n}}\left(1+\frac{x^\beta}{x}\right)^{\frac{1}{p_n}} \\
= \ & g(x+x^\beta)x^{\frac{1}{p_n}}\left(1+\frac{1}{p_n}\frac{x^\beta}{x} + f_{p_n}\left(\frac{x^\beta}{x}\right)\left(\frac{x^\beta}{x}\right)^2\right) \\
= \ & g(x+x^\beta)x^{\frac{1}{p_n}} + o\left(x^{\frac{1}{p_n}}\right) = \left(g(x+x^\beta) - g(x)\right)x^{\frac{1}{p_n}} + g(x)x^{\frac{1}{p_n}} \\
+ \ & o\left(x^{\frac{1}{p_n}}\right) = g(x)x^{\frac{1}{p_n}} + o\left(x^{\frac{1}{p_n}}\right),
\end{aligned} \tag{3}$$

since $\lim_{x\to\infty} g(x) = 1$.

On the other hand, if $s \geq 2$ is a positive integer we have (Lemma 1.2)

$$\left(x + x^{\beta}\right)^{\frac{1}{s}} = x^{\frac{1}{s}} \left(1 + \frac{x^{\beta}}{x}\right)^{\frac{1}{s}} = x^{\frac{1}{s}} \left(1 + \frac{1}{s}\frac{x^{\beta}}{x} + f_s\left(\frac{x^{\beta}}{x}\right)\left(\frac{x^{\beta}}{x}\right)^2\right)$$

$$= x^{\frac{1}{s}} + \frac{1}{s} x^{\frac{1}{s}+\beta-1} + f_s\left(\frac{x^{\beta}}{x}\right) x^{\frac{1}{s}+2\beta-2}. \tag{4}$$

Equation (1) gives

$$N(x + x^{\beta}) = \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1,\ p_{i_1} \cdots p_{i_k} < p_n} \left(x + x^{\beta}\right)^{\frac{1}{p_{i_1} \cdots p_{i_k}}}$$

$$+ \quad g(x + x^{\beta}) \left(x + x^{\beta}\right)^{\frac{1}{p_n}}. \tag{5}$$

Substituting (3) and (4) into (5) and using (1) we find that

$$N(x + x^{\beta}) = \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1,\ p_{i_1} \cdots p_{i_k} < p_n} \left(x + x^{\beta}\right)^{\frac{1}{p_{i_1} \cdots p_{i_k}}}$$

$$+ \quad g(x + x^{\beta}) \left(x + x^{\beta}\right)^{\frac{1}{p_n}} = \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1,\ p_{i_1} \cdots p_{i_k} < p_n} \left(x^{\frac{1}{p_{i_1} \cdots p_{i_k}}} \right.$$

$$+ \quad \left. \frac{1}{p_{i_1} \cdots p_{i_k}} x^{\frac{1}{p_{i_1} \cdots p_{i_k}}+\beta-1} + h_{p_{i_1} \cdots p_{i_k}}(x) x^{\frac{1}{p_{i_1} \cdots p_{i_k}}+2\beta-2} \right) + g(x) x^{\frac{1}{p_n}} + o\left(x^{\frac{1}{p_n}}\right)$$

$$= \quad N(x) + \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1,\ p_{i_1} \cdots p_{i_k} < p_n} \left(\frac{1}{p_{i_1} \cdots p_{i_k}} x^{\frac{1}{p_{i_1} \cdots p_{i_k}}+\beta-1} \right.$$

$$+ \quad \left. h_{p_{i_1} \cdots p_{i_k}}(x) x^{\frac{1}{p_{i_1} \cdots p_{i_k}}+2\beta-2} \right) + o\left(x^{\frac{1}{p_n}}\right). \tag{6}$$

where

$$h_{p_{i_1} \cdots p_{i_k}}(x) = f_{p_{i_1} \cdots p_{i_k}}\left(\frac{x^{\beta}}{x}\right)$$

That is,

$$N(x + x^{\beta}) = N(x) + \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1,\ p_{i_1} \cdots p_{i_k} < p_n}$$

$$\left(\frac{1}{p_{i_1} \cdots p_{i_k}} x^{\frac{1}{p_{i_1} \cdots p_{i_k}}+\beta-1} + h_{p_{i_1} \cdots p_{i_k}}(x) x^{\frac{1}{p_{i_1} \cdots p_{i_k}}+2\beta-2} \right) + o\left(x^{\frac{1}{p_n}}\right). \tag{7}$$

Note that among the numbers $p_{i_1} \ldots p_{i_k}$ are the primes (when $k = 1$, see (7)) $p_1 = 2$, $p_2 = 3$, ..., p_{n-1}. Consequently $p_1 = 2$ and $p_2 = 3$ are the least numbers $p_{i_1} \ldots p_{i_k}$. We wish eliminate all exponents in (7), namely

$$\frac{1}{p_{i_1} \cdots p_{i_k}} + \beta - 1,$$

and

$$\frac{1}{p_{i_1}\dots p_{i_k}} + 2\beta - 2,$$

except the exponent that correspond to $p_{i_1}\dots p_{i_k} = p_1 = 2$, namely

$$\frac{1}{2} + \beta - 1. \tag{8}$$

If we choose

$$\beta = \frac{1}{2} + \frac{1}{p_n} + \lambda, \tag{9}$$

where $0 < \lambda < \frac{1}{6}$ then (see (8) and (9))

$$\frac{1}{2} + \beta - 1 = \frac{1}{p_n} + \lambda > \frac{1}{p_n}, \tag{10}$$

$$\frac{1}{p_{i_1}\dots p_{i_k}} + \beta - 1 < \frac{1}{p_n}, \tag{11}$$

since $p_{i_1}\dots p_{i_k} \geq 3$. On the other hand, if $p_{i_1}\dots p_{i_k} \geq 3$ then (see (11))

$$\frac{1}{p_{i_1}\dots p_{i_k}} + 2\beta - 2 < \frac{1}{p_{i_1}\dots p_{i_k}} + \beta - 1 < \frac{1}{p_n}, \tag{12}$$

Besides if $p_{i_1}\dots p_{i_k} = p_1 = 2$ then

$$\frac{1}{2} + 2\beta - 2 < \frac{1}{p_n}, \tag{13}$$

since $p_n \geq 7$. Consequently if β satisfies (9) then equation (7) becomes (see (9), (10), (11), (12) and (13))

$$N(x + x^{\frac{1}{2}+\frac{1}{p_n}+\lambda}) = N(x) + \frac{1}{2}x^{\frac{1}{p_n}+\lambda} + o\left(x^{\frac{1}{p_n}}\right).$$

That is, equation (2). The lemma is proved. □

Theorem 2.2. *If $1/6 < \omega < 1/7 + 1/6$ is a fixed number then we have the following asymptotic formula*

$$N(x + x^{\frac{1}{2}+\omega}) = N(x) + \frac{1}{2}x^{\omega} + o\left(x^{\frac{1}{p_n}}\right),$$

where p_n is the greatest prime that appear in the solutions (p_n, λ) to the equation

$$\frac{1}{p_n} + \lambda = \omega \qquad (n \geq 4)$$

Proof. If $1/6 < \omega < 1/7 + 1/6$ then the equation

$$\frac{1}{p_n} + \lambda = \omega \qquad (n \geq 4)$$

has a finite number of solutions (p_n, λ). Consequently equation (2) becomes

$$N(x + x^{\frac{1}{2}+\omega}) = N(x) + \frac{1}{2}x^{\omega} + o\left(x^{\frac{1}{p_n}}\right),$$

where p_n is the greatest prime in this finite number of solutions. The theorem is proved.

Theorem 2.3. *If* $0 < \omega \leq 1/6$ *is a fixed number then we have the following asymptotic formula*

$$N(x + x^{\frac{1}{2}+\omega}) = N(x) + \frac{1}{2}x^{\omega} + o\left(x^{\alpha}\right),$$

for all $0 < \alpha < \omega$.

Proof. If $0 < \omega \leq 1/6$ then the equation

$$\frac{1}{p_n} + \lambda = \omega \qquad (n \geq 4)$$

has infinite solutions (p_n, λ), where $n \geq n_0$. Consequently equation (2) becomes

$$N(x + x^{\frac{1}{2}+\omega}) = N(x) + \frac{1}{2}x^{\omega} + o\left(x^{\alpha}\right),$$

for all $0 < \alpha < \omega$. The theorem is proved. □

Theorem 2.4. *If* $0 < \epsilon < 1/7 + 1/6$ *is a fixed number we have the following asymptotic formula,*

$$N(x + x^{\frac{1}{2}+\epsilon}) = N(x) + \frac{1}{2}x^{\epsilon} + o\left(x^{\epsilon}\right). \tag{14}$$

Proof. Equation (2) can be written in the more weak form,

$$N(x + x^{\frac{1}{2}+\frac{1}{p_n}+\lambda}) = N(x) + \frac{1}{2}x^{\frac{1}{p_n}+\lambda} + o\left(x^{\frac{1}{p_n}+\lambda}\right).$$

If we write $\epsilon = (1/p_n) + \lambda$ then $0 < \epsilon < 1/7 + 1/6$. The theorem is proved. □

Theorem 2.5. *If* $\epsilon > 0$ *then we have the following limit,*

$$\lim_{x \to \infty} \left(N(x + x^{\frac{1}{2}+\epsilon}) - N(x)\right) = \infty. \tag{15}$$

If the exponent is less than or equal to $\frac{1}{2}$ *this limit is false.*

Proof. Limit (15) is an immediate consequence of equation (14). If the exponent is $1/2$ then we have the difference $N(x + x^{\frac{1}{2}}) - N(x)$. It is well-known (Theorem 3.1, [1]) that there exist infinite $x = n^2$ such that $N(n^2 + n) - N(n^2) = 0$. The theorem is proved. □

References

[1] Jakimczuk, R., On the distribution of perfect powers, *Journal of Integer Sequences*, Vol. 14, 2011, Article 11.8.5.