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Some arithmetic properties of an analogue of Möbius function

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Abstract: Some properties and applications of an analogue of Möbius function are studied in the paper titled, "Some properties and application of a new arithmetic function in analytic number theory". In this paper, some additional properties of this new arithmetic function connecting with familiar arithmetic functions such as Möbius function, Euler totient function, etc., are given.
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1 Introduction

An analogue of Möbius function υ_p is introduced in the paper titled, "Some properties and application of a new arithmetic function in analytic number theory" [3]. For any two positive integers p and n, it is defined as follows

$$\upsilon_p(n) = \begin{cases} 1 & \text{If } n = 1 \\ 2^{r-1} & \text{If } n = p^r,\ r \in N,\ p > 1 \\ (-1)^k & \text{If either } p \nmid n \text{ or } p = 1 \text{ and } n \text{ is} \\ & \text{square-free with } k \text{ distinct primes} \\ (-1)^k 2^{r-1} & \text{If } n = p^r m,\ r \in N,\ p \nmid m \text{ and } m \text{ is} \\ & \text{square-free with } k \text{ distinct primes} \\ 0 & \text{Otherwise.} \end{cases} \tag{1}$$

The following are some simple properties of υ_p for prime p relating with Möbius function.

1. If $p \nmid n$ and n is square free, then $\upsilon_p(n) = \mu(n)$.
2. If $n = p^r m, p \nmid m$ and m is square free, then $\upsilon_p(n) = 2^{r-1}\mu(m)$.
3. In particular $\upsilon_1(n) = \mu(n)$, $n \in N$.

Another remarkable property connecting with the function $\delta_p(n)$ for prime p is as follows

$$\sum_{d|n} \delta_p(n/d)\upsilon_p(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases} \tag{2}$$

For any two integers n and p, $\delta_p(n)$ is defined as follows

$$\delta_p(n) = \begin{cases} -1 & \text{If } p|n \text{ and } p > 1 \\ 1 & \text{If either } p \nmid n \text{ or } p = 1\ . \end{cases} \tag{3}$$

Some applications of this new arithmetic function υ_p connecting with infinite products and partition of an integer are studied in Ref[3]. In this present study, some arithmetic properties of υ_p are derived with familiar arithmetic function such as Möbius function, Euler totient function etc,.

2 Arithmetic properties of analogue of Möbius function υ_p

Theorem 2.1. *Let f be a multiplicative function, and let p a prime and $k \in N$. If $p \nmid m$, then*

$$\sum_{d|p^k m} f(d) = \sum_{d|m} f(d) \times \sum_{j=0}^{k} f(p^j). \tag{4}$$

Proof.

$$\sum_{d|p^k m} f(d) = \sum_{j=0}^{k} \sum_{d|mp^j} f(d)$$
$$= \sum_{j=0}^{k} \sum_{d/p^j|m} f(d).$$

Let $d/p^j = d_1$. Then

$$\sum_{d|p^k m} f(d) = \sum_{j=0}^{k} \sum_{d_1|m} f(d_1 p^j).$$

Since f is multiplicative, after simplification, gives (2.1). □

Theorem 2.2. *Let p be a prime and $p \nmid m$. Then*

$$\sum_{d|p^k m} \upsilon_p(d) = \begin{cases} 2^k & m = 1 \\ 0 & m \neq 1. \end{cases} \tag{5}$$

Proof. Using Theorem 2.1,

$$\sum_{d|p^k m} \upsilon_p(d) = \sum_{d|m} \upsilon_p(d) \times \sum_{j=0}^{k} \upsilon_p(p^j).$$

From the definition of $\upsilon_p(p^j)$, it is clear that $\upsilon_p(p^j) = 2^{j-1}$. Then

$$\begin{aligned}\sum_{d|p^k m} \upsilon_p(d) =& \sum_{d|m} \mu(d) \times \left(1 + \sum_{j=1}^{k} 2^{j-1}\right). \\ =& 2^k \sum_{d|m} \mu(d).\end{aligned}$$

If $m = 1$, then $\upsilon_p(p^k m) = 2^k$. Since $\sum_{d|m} \mu(d) = 0$[1, p. 25], $\upsilon_p(p^k m) = 0$ for $m \neq 1$. This completes the theorem. □

Remark 2.3. It is clear that from the definition of υ_p, if $p \nmid m$ then $\upsilon_p(m) = \mu(m)$.

Theorem 2.4. *Let ϕ be Euler totient function, and let p be prime and $p|m$*

$$\sum_{d|p^k m} \frac{\upsilon_p(d)}{d} = \frac{\phi(m)}{m} \left[1 + \frac{p^k - 2^k}{p^k(p-2)}\right]. \tag{6}$$

Proof. Let $f(n) = \upsilon_p(n)/n$ in Theorem 2.1. Then

$$\begin{aligned}\sum_{d|p^k m} \frac{\upsilon_p(d)}{d} =& \sum_{d|m} \frac{\upsilon_p(d)}{d} \sum_{j=0}^{k} \frac{\upsilon_p(p^j)}{p^j} \\ =& \sum_{d|m} \frac{\mu(d)}{d} \left[1 + \sum_{j=1}^{k} \frac{2^{j-1}}{p^j}\right].\end{aligned}$$

Using Remark 2.3 and the identity $\sum_{d|m} \mu(d)/d = \phi(m)/m$[1, p. 26], after simplification, gives (2.3). □

Theorem 2.5. *Let ϕ^{-1} is inverse of Euler totient function ϕ with respect to Dirichlet convolution and let p be prime and $p \nmid m$*

$$\sum_{d|p^k m} \upsilon_p(d)d = \phi^{-1}(m) \left[1 + p\frac{2^k p^k - 1}{(2p-1)}\right]. \tag{7}$$

Proof. Let $f(n) = \upsilon_p(n)d$ in Theorem 2.1. Then

$$\begin{aligned}\sum_{d|p^k m} \upsilon_p(d)d =& \sum_{d|m} \upsilon_p(d)d \sum_{j=0}^{k} \upsilon_p(p^j)p^j \\ =& \sum_{d|m} \mu(d)d \left[1 + \sum_{n=1}^{k} 2^{n-1}p^n\right].\end{aligned}$$

Since $\sum_{d|m} \mu(d)d = \phi^{-1}(m)$[1, p. 37], after simplification, gives (2.4). □

Theorem 2.6. *Let p be prime and $p \nmid m$. Then*

$$\sum_{d|p^k m} \upsilon_p(d)^2 = 2^{v(m)} \left[1 + \frac{4^k - 1}{3}\right]. \tag{8}$$

Where $v(1) = 0$, if $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, then $v(n) = k$.

Proof. Let $f(n) = \upsilon_p(n)$ in Theorem 2.1. Then

$$\begin{aligned}\sum_{d|p^k m} \upsilon_p(d)^2 &= \sum_{d|m} \upsilon_p(d)^2 \sum_{j=0}^{k} \upsilon_p(p^j)^2 \\ &= \sum_{d|m} \mu(d)^2 \left[1 + \sum_{n=1}^{k} 2^{2n-2}\right].\end{aligned}$$

Since $\sum_{d|m} \mu(d)^2 = 2^{v(m)}$[1, p. 45], after simplification, gives (2.5). □

3 Generalization of Theorem 2.1

Theorem 3.1. *Let f be a multiplicative function, and let $p_1, p_2, \dots p_k$ are k any distinct prime numbers and $a_1, a_2 \dots, a_k \in N$. If each $p_j \nmid m$, $j \in N$, then*

$$\sum_{d|p_1^{a_1} p_2^{a_2} \dots p_k^{a_k} m} f(d) = \sum_{d|m} f(d) \times \sum_{j_1=0}^{a_1} f(p_1^{j_1}) \times \dots \times \sum_{j_k=0}^{a_k} f(p_1^{j_k}). \tag{9}$$

Proof. The left hand side of (3.1) can be written using Theorem 2.1 as follows

$$\sum_{d|p_1^{a_1} p_2^{a_2} \dots p_k^{a_k} m} f(d) = \sum_{d|p_2^{a_2} \dots p_k^{a_k} m} f(d) \times \sum_{j_1=0}^{a_1} f(p_1^{j_1}).$$

Repeating this process k times, gives (3.1). This completes the theorem. □

Corollary 3.2. *If f be an completely multiplicative function, then*

$$\sum_{d|p_1^{a_1} p_2^{a_2} \dots p_k^{a_k} m} f(d) = \sum_{d|m} f(d) \times \prod_{i=1}^{k} \frac{f(p_i)^{a_i+1} - 1}{f(p_i) - 1}. \tag{10}$$

Proof. Using Theorem 3.1, gives

$$\sum_{d|p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}} f(d) = \sum_{d|m} f(d) \times \sum_{j_1=0}^{a_1} f(p_1)^{j_1} \times \dots \times \sum_{j_k=0}^{a_k} f(p_k)^{j_k}. \tag{11}$$

Since f is completely multiplicative, for each i

$$\sum_{j_i=0}^{a_i} f(p_i)^{j_i} = \frac{f(p_i)^{a_i+1} - 1}{f(p_i) - 1}. \tag{12}$$

Using (3.4) in (3.3) and after simplification, gives (3.2). This completes the corollary. □

Example 3.3. Let $f(n) = n^s$, where $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ in (3.2). Then

$$\sigma_s(n) = \prod_{i=1}^{k} \frac{p_i^{s(a_i+1)} - 1}{p_i^s - 1}, \tag{13}$$

where σ_s is divisor function.

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