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Nesterenko-like rational function, useful to prove the Apéry's theorem

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Abstract: In this paper, a brief introduction to the Apéry's result and to the so called phenomenon of Apéry's is given. Here, a modification of the Nesterenko's rational function, from which new diophantine approximations to $\zeta(3)$ are deduced, is presented. Moreover, as a consequence we deduce the corresponding Apéry-Like recurrence relation as well as a new continued fraction expansion and a new series expansion for $\zeta(3)$.
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1 Introduction

One of the more interesting open problems into the number theory has to do with the arithmetical nature of the values of the Riemann zeta function [16]

$$\zeta(k) \equiv \sum_{n\geq 1} \frac{1}{n^k} = \frac{(-1)^{k-1}}{(k-1)!} \int_0^1 \frac{\log^{k-1} x}{1-x} dx, \tag{1}$$

in the positive integers $k \in \mathbb{N}\setminus\{1\}$. As it is well known, at the Journees Arithmetiques held at Marseille-Luminy in June 1978, Roger Apéry [5, 11, 28] gave an elementary proof of the following result, credited as Apéry's theorem.

Theorem 1.1. *The number $\zeta(3)$ is irrational.*

However due to the complexity of the Apéry's proof, there was a general disagreement amongst the mathematicians present there as to the validity of the proof presented. Two months later a complete exposition of the proof was presented at the International Congress of Mathematicians in Helsinki in August 1978 by H. Cohen. This proof based on the lecture by Apéry but also contained some ideas of Cohen and Don Zaiger [11, 28].

Theorem 1.2. (Criterion for irrationality) *If there is a $\delta > 0$ and a sequence $(v_n/u_n)_{n\geq 0}$ of rational numbers such that $v_n/u_n \neq x$ and*

$$\left| x - \frac{v_n}{u_n} \right| < \frac{1}{u_n^{1+\delta}}, \quad n = 0, 1, \ldots,$$

then x is irrational.

From now a sketch of the Apéry's proof will take place. The same one had as a fundamental ingredient the following recurrence relation [6, 12, 28]

$$(n+2)^3 y_{n+2} - (2n+3)(17n^2+51n+39) y_{n+1} + (n+1)^3 y_n = 0, \quad n \geq 0, \tag{2}$$

which is satisfied by the numerators a_n and denominators b_n of the diophantine approximations to $\zeta(3)$ with the initial conditions

$$a_0 = 0, \quad a_1 = 6, \quad b_0 = 1, \quad b_1 = 5,$$

or by the explicit representation of the sequences in question

$$b_n \equiv \sum_{0\leq k\leq n} \binom{n+k}{k}^2 \binom{n}{k}^2 \quad \text{and} \quad a_n \equiv \sum_{0\leq k\leq n} \binom{n+k}{k}^2 \binom{n}{k}^2 \gamma_{n,k}, \tag{3}$$

where

$$\gamma_{n,k} = \sum_{1\leq j\leq n} \frac{1}{j^3} + \sum_{1\leq j\leq k} \frac{(-1)^{j-1}}{2j^3} \binom{n+j}{j}^{-1} \binom{n}{j}^{-1}.$$

Observe that, from (2) we deduce that

$$a_n b_{n-1} - a_{n-1} b_n = \frac{6}{n^3}.$$

This leads to an example of a non-simple continued fraction expansion

$$\zeta(3) = \frac{6\,|}{|\,5} - \frac{1\,|}{|\,117} - \frac{64\,|}{|\,535} - \cdots - \frac{n^6 \quad |}{|\,(2n+1)(17n^2+17n+5)} - \cdots,$$

Then, seeing that $\zeta(3) - a_0/b_0 = \zeta(3)$, it is induced that

$$\left| \zeta(3) - \frac{a_n}{b_n} \right| = \sum_{k\geq n+1} \frac{6}{k^3 b_{k-1} b_k} = \mathcal{O}\left(b_n^{-2}\right).$$

It is easy to verify by the recurrence relation (2) and Poincaré's theorem [22, 23] that

$$b_n = \mathcal{O}(\varpi^n), \quad \text{where} \quad \varpi = \left(\sqrt{2}+1\right)^4.$$

Moreover, by the prime number theorem, it can be shown that

$$l_n \equiv \prod_{p \leq n} p^{\left[\frac{\log n}{\log p}\right]} \leq \prod_{p \leq n} n = \mathcal{O}\left(e^{(1+\epsilon)n}\right), \quad \forall\, \epsilon > 0. \tag{4}$$

Therefore, setting $v_n = 2a_n l_n^3 \in \mathbb{Z}$ and $u_n = 2b_n l_n^3 \in \mathbb{Z}$, we obtain $u_n = \mathcal{O}\left(\varpi^n e^{3n}\right)$ and

$$\left|\zeta(3) - \frac{v_n}{u_n}\right| = \mathcal{O}\left(u_n^{-(1+\delta)}\right),$$

where

$$\delta = \frac{\log \alpha - 3}{\log \alpha + 3} = 0.080529\ldots > 0,$$

which, proves the Apéry's theorem, by virtue of criterion for irrationality. Apéry's irrationality proof of $\zeta(3)$ operates with the faster convergent series

$$\zeta(3) = \frac{5}{2} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}},$$

first obtained by A. A. Markov in 1890 [18].

The aforementioned result, in a beginning somewhat polemic, inspired to several mathematicians to construct different methods to explain the irrationality of aforesaid constant [6, 7, 8, 10, 19, 20, 24, 26, 27, 29]. Surprisingly, these methods conduce to the same sequences of diophantine approximations (3) to $\zeta(3)$ (named Apéry's approximants or Apéry's diophantine approximations). This fact names Apéry's phenomenon. In several papers Apéry's phenomenon it has been interpreted in different ways. For example: shortly after Apéry announced his proof, Beukers produced an elegant and entirely different proof of the irrationality of $\zeta(2)$ and $\zeta(3)$. In the case of $\zeta(2)$, Beukers in [7, 8] considers a double integrals defined by

$$\int_0^1 \int_0^1 \frac{(1-y)^n \mathcal{L}_n(x)}{1-xy} dxdy = \theta_n \zeta(2) - \vartheta_n,$$

where $n \in \mathbb{N}$ and $\mathcal{L}_n(x)$ is the Legendre-type polynomial, orthogonal with respect to the Lebesgue measure on $(0,1)$, given by

$$\mathcal{L}_n(z) = \frac{1}{n!} \frac{d^n}{dz^n} z^n (1-z)^n = \sum_{0 \leq k \leq n} (-1)^k \binom{n+k}{k} \binom{n}{k} z^k,$$

and θ_n, $\vartheta_n \in \mathbb{Q}$, for all n. An estimation of the latter linear form shows that it tends to 0 as n approaches infinity fast enough to yield the irrationality of $\zeta(2)$. Beukers also gives an analogous argument for the case of $\zeta(3)$ [7, 8, 20] by using the following triple integral instead

$$\begin{aligned} b_n \zeta(3) - a_n &= -\int_0^1 \int_0^1 \frac{\log xy}{1-xy} \mathcal{L}_n(x) \mathcal{L}_n(y)\, dxdy \\ &= \int_0^1 \int_0^1 \int_0^1 \frac{\left(xyz(1-x)(1-y)(1-z)\right)^n}{\left(1-(1-xy)z\right)^{n+1}} dxdydz. \end{aligned} \tag{5}$$

Thus, Beukers shows that previous expression coincides with $\mathcal{O}\left(\varpi^{-n}\right)$, which proves the Apéry's theorem.

In [9], Beukers considered a rational approximation problem in an intent to formulate Apéry's proof more natural and introduced the rational function

$$\mathcal{R}_n(z) \equiv \frac{(n-z+1)_n^2}{(-z)_{n+1}^2}, \tag{6}$$

from which by computing the partial fraction expansion, deduces the Apéry's rational approximants. Here, $(\cdot)_n$ denotes the Pochhammer symbol [13], also called the shifted factorial, defined by

$$(z)_n \equiv \prod_{0\le j\le n-1} (z+j), \quad n \ge 1, \quad (z)_0 = 1,$$

which in terms of the gamma function is given by

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}, \quad n = 0, 1, 2, \ldots$$

Sorokin, in virtue of a approximation problem, obtained Apéry's sequences as well as Beukers's error-term sequence (5) [26, 29]. After this, Nesterenko in [20], inspired by Gutnik's work [14] considered the following modification of the Beukers's rational function (6)

$$\mathcal{N}_n(z) \equiv \frac{(-z)_n^2}{(z+1)_{n+1}^2}, \tag{7}$$

and proved the following expression for the error-term sequence

$$b_n\zeta(3) - a_n = -\sum_{k\ge 0} \frac{d}{dk}\mathcal{N}_n(k) = \frac{1}{2\pi\mathrm{i}}\int_L \mathcal{N}_n(\nu)\left(\frac{\pi}{\sin\pi\nu}\right)^2 d\nu,$$

where

$$\frac{d}{dz}\mathcal{N}_n(z) = 2\mathcal{N}_n(z)\left(\sum_{0\le k\le n-1}\frac{1}{t-k} - \sum_{1\le k\le n+1}\frac{1}{t+k}\right),$$

and L is the vertical line Re $z = C$, $0 < C < n+1$, oriented from top to bottom. Indeed, the use of Laplace's method allowed him to estimate the above contour integral (7) yielding the behavior $\mathcal{O}(\varpi^{-n})$. Moreover, he proposed the following continued fraction expansion

$$2\zeta(3) = 2 + \frac{1\,|}{|\,2} + \frac{2\,|}{|\,4} + \frac{1\,|}{|\,3} + \frac{4\,|}{|\,2} + \frac{2\,|}{|\,4} + \frac{6\,|}{|\,6} + \frac{4\,|}{|\,5} + \cdots,$$

where the numerators a_n, $n \ge 2$, and denominators b_n, $n \ge 1$, are definded by

$$\begin{aligned} b_{4k+1} &= 2k+2, \quad a_{4k+1} = k(k+1), \quad b_{4k+2} = 2k+4, \\ a_{4k+2} &= (k+1)(k+2), \quad b_{4k+3} = 2k+3, \\ a_{4k+3} &= (k+1)^2, \quad b_{4k} = 2k, \quad a_{4k} = (k+1)^2. \end{aligned}$$

The purpose of this paper is to present a modification of the Nesterenko's rational function (7) defined by

$$\mathcal{F}_n(z) \equiv \frac{(-z)_{n-1}^2(n-z-1)}{(z+1)_{n+1}^2}. \tag{8}$$

From which we deduce new rational approximants that prove Apéry's theorem. This modification, consists in deleting one of the zeroes of the rational function (7), in particular $z = n-1$. Moreover, as a consequence we deduce the corresponding Apéry-Like recurrence relation for these rational approximants as well as a new continued fraction expansion for $\zeta(3)$. Also, we show a new series expansion for $\zeta(3)$, inspired by Arvesú's work in [6].

2 Apéry-like recurrence relation

Lemma 2.1. *The following relation is valid*

$$r_n = -2^{-1} \sum_{z \geq 0} \frac{d}{dz} \mathcal{F}_n(z) = q_n \zeta(3) - p_n, \quad n = 0, 1, \ldots, \tag{9}$$

being

$$q_n = \sum_{0 \leq k \leq n} b_k^{(n)} \quad \text{and} \quad p_n = \sum_{1 \leq k \leq n} b_k^{(n)} H_k^{(3)} + 2^{-1} \sum_{1 \leq k \leq n} a_k^{(n)} H_k^{(2)}, \tag{10}$$

where the coefficients $b_k^{(n)}$ and $a_k^{(n)}$ are given in (11)-(12), respectively. Moreover, $nq_n \in \mathbb{Z}$ and $2nl_n^3 p_n \in \mathbb{Z}$. Here, we denote for $H_k^{(r)}$ to the harmonic number k of order r.

In fact, let us expand the function $\mathcal{F}_n(z)$ on the sum of partial fractions

$$\mathcal{F}_n(z) = \sum_{0 \leq k \leq n} \frac{a_k^{(n)}}{z+k+1} + \sum_{0 \leq k \leq n} \frac{b_k^{(n)}}{(z+k+1)^2},$$

where

$$\begin{aligned} b_k^{(n)} &= (z+k+1)^2 \mathcal{F}_n(z) \Big|_{z=-k-1}, \quad k = 0, \ldots, n, \\ &= \binom{n+k}{k}^2 \binom{n}{k}^2 (n+k)^{-1} \\ &= \frac{1}{n} \binom{n+k-1}{k}^2 \binom{n}{k}^2 + \frac{1}{n} \binom{n+k-1}{k}^2 \binom{n-1}{k-1} \binom{n}{k}, \end{aligned} \tag{11}$$

and

$$\begin{aligned} a_k^{(n)} &= \frac{d}{dt} \left[(z+k+1)^2 \mathcal{F}_n(z) \right] \Big|_{z=-k-1}, \quad k = 0, \ldots, n, \\ &= 2b_k^{(n)} \left[2H_k - H_{n+k-1} - H_{n-k} - 2^{-1} (n+k)^{-1} \right]. \end{aligned} \tag{12}$$

Since $\mathcal{F}_n(z) = \mathcal{O}(z^{-2})$ as $z \to \infty$, then we infer

$$\sum_{0 \leq k \leq n} a_k^{(n)} = \sum_{0 \leq k \leq n} \operatorname*{Res}_{z=-k-1} \mathcal{F}_n(z) = -\operatorname*{Res}_{z=\infty} \mathcal{F}_n(z) = 0.$$

Thus, we have

$$\begin{aligned}
r_n &= \sum_{z\geq 0}\sum_{0\leq k\leq n}\frac{b_k^{(n)}}{(z+k+1)^3}+2^{-1}\sum_{z\geq 0}\sum_{0\leq k\leq n}\frac{a_k^{(n)}}{(z+k+1)^2}\\
&= \sum_{0\leq k\leq n}\sum_{l\geq k+1}\frac{b_k^{(n)}}{l^3}+2^{-1}\sum_{0\leq k\leq n}\sum_{l\geq k+1}\frac{a_k^{(n)}}{l^2}\\
&= \sum_{0\leq k\leq n} b_k^{(n)}\left(\sum_{l\geq 1}-\sum_{1\leq l\leq k}\right)\frac{1}{l^3}+2^{-1}\sum_{0\leq k\leq n} a_k^{(n)}\left(\sum_{l\geq 1}-\sum_{1\leq l\leq k}\right)\frac{1}{l^2}\\
&= \sum_{0\leq k\leq n} b_k^{(n)}\sum_{l\geq 1}\frac{1}{l^3}-\sum_{1\leq k\leq n} b_k^{(n)}\sum_{1\leq l\leq k}\frac{1}{l^3}-2^{-1}\sum_{1\leq k\leq n} a_k^{(n)}\sum_{1\leq l\leq k}\frac{1}{l^2},
\end{aligned}$$

which coincides with (9) by considering the expressions given in (10). Clearly, from the expressions (11)–(12) it follows that $nb_k^{(n)}\in\mathbb{Z}$ and $nl_na_k^{(n)}\in\mathbb{Z}$, $k=0,\ldots,n$. Therefore, using

$$l_n^j\sum_{i=1}^{k}\frac{1}{i^j}\in\mathbb{Z},\quad k=0,1,\ldots,n,\quad j\in\mathbb{Z}^+,$$

we deduce that $nq_n\in\mathbb{Z}$ and $2nl_n^3p_n\in\mathbb{Z}$ as required. The lemma is completely proved.

Before continuing with the following result let us define

$$\rho_{1,n}\equiv 2(n+2)(384n^8+2592n^7+7516n^6+12228n^5 + 12255n^4+7838n^3+3166n^2+730n+73),$$

$$\rho_{2,n}\equiv 2(288n^7+1776n^6+4682n^5+6677n^4 + 5390n^3+2448n^2+628n+69),$$

$$\rho_{3,n}\equiv 2\left(96n^5+360n^4+478n^3+285n^2+88n+11\right),$$

and

$$\frac{\mathcal{S}_n(z)}{(k-n)(k-n+1)}\equiv\frac{\rho_{2,n}z+(2n^2+5n+6)\rho_{3,n}z^2+\rho_{3,n}z^3-\rho_{1,n}}{(z+n+2)^2(n+1)^2},$$

which is the output of the so-called algorithm of creative telescoping due to W. Gosper and D. Zeilberger [1, 2, 3, 4, 25]. The algorithm is realised in different computer algebra systems, in particular, in Maple and Mathematica.

Proposition 2.2. *The sequences $(p_n)_{n\geq 1}$, $(q_n)_{n\geq 1}$ and $(r_n)_{n\geq 1}$ verify the following Apéry-like recurrence relation*

$$\begin{aligned}
(n+2)^4\left(24n^3+30n^2+16n+3\right)y_{n+2} &\\
-4(n+1)(204n^6+1173n^5+2668n^4+3065n^3 &\\
+1905n^2+634n+86y_{n+1})y_{n+1} &\\
+n^4\left(24n^3+102n^2+148n+73\right)y_n=0,\quad n\geq 1. &\quad (13)
\end{aligned}$$

Proof. In fact, firstly let us prove that $(r_n)_{n\geq 0}$ verify the previous Apéry-like recurrence relation (13). For such purpose, it is enough to check

$$\begin{aligned}(n+2)^4 &\left(24n^3+30n^2+16n+3\right)\mathcal{F}_{n+2}\\ &-4(n+1)(204n^6+1173n^5+2668n^4+3065n^3\\ &+1905n^2+634n+86y_{n+1})\mathcal{F}_{n+1}\\ &+n^4\left(24n^3+102n^2+148n+73\right)\mathcal{F}_n\\ &=\mathcal{S}_n(z+1)\,\mathcal{F}_n(z+1)-\mathcal{S}_n(z)\,\mathcal{F}_n(z)\,,\quad n\geq 1. \quad (14)\end{aligned}$$

Thus, we have

$$\begin{aligned}(n+2)^4 &\left(24n^3+30n^2+16n+3\right)r_{n+2}\\ &-4(n+1)(204n^6+1173n^5+2668n^4+3065n^3\\ &+1905n^2+634n+86y_{n+1})r_{n+1}\\ &+n^4\left(24n^3+102n^2+148n+73\right)r_n\\ &=-2^{-1}\sum_{z\geq 0}\frac{d}{dz}\left[\mathcal{S}_n(z+1)\,\mathcal{F}_n(z+1)-\mathcal{S}_n(z)\,\mathcal{F}_n(z)\right]\\ &=-2^{-1}\mathcal{S}_n'(0)\,\mathcal{F}_n(0)-2^{-1}\mathcal{S}_n(0)\,\mathcal{F}_n'(0)=0.\end{aligned}$$

Since $\mathcal{F}_n(0)=\mathcal{F}_n'(0)=0$ for all $n\geq 1$.

For the sequence $(q_n)_{n\geq 1}$, observe that $b_k^{(n)}=0$ for $k<0$ and $k>n$. In consequence, using (14) we have

$$\begin{aligned}(n+2)^4 &\left(24n^3+30n^2+16n+3\right)q_{n+2}\\ &-4(n+1)(204n^6+1173n^5+2668n^4+3065n^3\\ &+1905n^2+634n+86y_{n+1})q_{n+1}\\ &+n^4\left(24n^3+102n^2+148n+73\right)q_n\\ &=\sum_{k\in\mathbb{Z}}\left[\tilde{\mathcal{S}}_n(z+1)-\tilde{\mathcal{S}}_n(z)\right](z+k+1)^2\Big|_{z=-k-1},\end{aligned}$$

where $\tilde{\mathcal{S}}_n(z)=\mathcal{S}_n(z)\,\mathcal{F}_n(z)$ and

$$\begin{aligned}\sum_{k\in\mathbb{Z}}&\left[\tilde{\mathcal{S}}_n(z+1)-\tilde{\mathcal{S}}_n(z)\right](z+k+1)^2\Big|_{z=-k-1}=\\ &\sum_{k\in\mathbb{Z}}\left[\tilde{\mathcal{S}}_n(j-k)-\tilde{\mathcal{S}}_n(j-k-1)\right]j^2\Big|_{j=0}=0.\end{aligned}$$

Finally, the sequence $(p_n=q_n\zeta(3)-r_n)_{n\geq 1}$ satisfies the Apéry's recurrence relation (2) as a linear combination of the sequences $(q_n)_{n\geq 1}$ and $(r_n)_{n\geq 1}$.

The proposition is completely proved. □

Notice that, from the Proposition 2.2 we have that the characteristic equation for (13) is $t^2-34t+1$ and their zeros are $t_1=\varpi$, and $t_2=\varpi^{-1}$ respectively. Hence, from Poincaré's theorem [22, 23] it's has the behavior $q_n = \mathcal{O}(\varpi^n)$ and $r_n = \mathcal{O}(\varpi^{-n})$, as n goes to infinity, for the two linearly independent solutions, respectively. Then, assuming that $\zeta(3) = p/q$, where $p, q \in \mathbb{Z}^+$, we have that $2qnl_n^3 r_n = 2pnl_n^3 q_n - 2qnl_n^3 p_n$, is an integer distinct from zero. Therefore, by the prime numbers theorem we have (4) and as a consequence we deduce that $1 \leq 2qnl_n^3 |r_n| = \mathcal{O}(l_n^3 \varpi^{-n})$, wich is a contradiction, because $e^3 \varpi^{-1} = 0,591263\ldots < 1$. Thus, the Apéry's theorem 1.1 is proven.

Theorem 2.3. *Let $(p_n)_{n\geq -1}$ and $(q_n)_{n\geq -1}$ be two sequences of numbers such that $q_{-1} = 0$, $p_{-1} = q_0 = 1$ and $p_n q_{n-1} - p_{n-1} q_n \neq 0$ for $n = 0, 1, 2, \ldots$. Then there exists a unique irregular continued fraction*

$$a_0 + \frac{b_1 \mid}{\mid a_1} + \frac{b_2 \mid}{\mid a_2} + \frac{b_3 \mid}{\mid a_3} + \cdots + \frac{b_n \mid}{\mid a_n} + \cdots, \tag{15}$$

whose n-th numerator is p_n and n-th denominator is q_n, for each $n \geq 0$. More precisely (see [15, p. 31])

$$a_0 = p_0, \quad a_1 = q_1, \quad b_1 = p_1 - p_0 q_1,$$

$$a_n = \frac{p_n q_{n-2} - p_{n-2} q_n}{p_{n-1} q_{n-2} - p_{n-2} q_{n-1}}, \quad b_n = \frac{p_{n-1} q_n - p_n q_{n-1}}{p_{n-1} q_{n-2} - p_{n-2} q_{n-1}}, \quad n = 0, 1, 2, \ldots$$

Theorem 2.4. *Two irregular continued fractions*

$$a_0 + \frac{b_1 \mid}{\mid a_1} + \frac{b_2 \mid}{\mid a_2} + \frac{b_3 \mid}{\mid a_3} + \cdots + \frac{b_n \mid}{\mid a_n} + \cdots, \quad a_0' + \frac{b_1' \mid}{\mid a_1'} + \frac{b_2' \mid}{\mid a_2'} + \frac{b_3' \mid}{\mid a_3'} + \cdots + \frac{b_n' \mid}{\mid a_n'} + \cdots,$$

are equivalent if and only if there exists a sequence of non-zero $(c_n)_{n\geq 0}$ with $c_0 = 1$ such that (see [15, p. 31])

$$a_n' = c_n a_n, \quad n = 0, 1, 2, \ldots, \quad b_n' = c_n c_{n-1} b_n, \quad n = 1, 2, \ldots \tag{16}$$

Then, using the previous theorem we deduce the following result.

Theorem 2.5. *The following irregular continued fraction expansion for $\zeta(3)$ is verify*

$$\zeta(3) = \frac{7 \mid}{\mid 6} + \frac{-146 \mid}{\mid \ 827} + \frac{-38864 \mid}{\mid \ \mathcal{Q}_3} + \frac{\mathcal{P}_4 \mid}{\mid \mathcal{Q}_4} + \cdots + \frac{\mathcal{P}_n \mid}{\mid \mathcal{Q}_n} + \cdots,$$

where

$$\mathcal{P}_n = -(n-2)^4 (n-1)^4 \left(24n^3 - 186n^2 + 484n - 423\right) \times \left(24n^3 - 42n^2 + 28n - 7\right),$$

and

$$\mathcal{Q}_n = 4(n-1) \times \left(204n^6 - 1275n^5 + 3178n^4 - 3999n^3 + 2667n^2 - 910n + 126\right).$$

3 New series expansion for $\zeta(3)$

We define

$$\varphi_n(z) \equiv A_n(z) - B_n(z)\log z \quad \text{and} \quad \psi_n(z) \equiv \varphi_n(z)\log z,$$

where $A_n(z)$ and $B_n(z)$ polynomials of degree exactly n defined by

$$A_n(z) \equiv \sum_{0\le k\le n} a_k^{(n)} z^n \quad \text{and} \quad B_n(z) \equiv \sum_{0\le k\le n} b_k^{(n)} z^n.$$

Thus, applying the identity

$$\int_0^1 x^i \log^j x dx = (-1)^j j! \frac{1}{(i+1)^{j+1}}, \tag{17}$$

we have

$$\begin{aligned} \mathcal{F}_n(z) &= \int_0^1 x^z \varphi_n(x)\, dx, \\ \mathcal{G}_n(z) &= \frac{d}{dz}\mathcal{F}_n(z) = \int_0^1 x^z \psi_n(x)\, dx. \end{aligned} \tag{18}$$

Then, from (8) and (18) we induce

$$\begin{aligned} &\int_0^1 x^k \varphi_n(x)\, dx = 0, \quad k = 0, \dots, n-1, \\ &\int_0^1 x^k \psi_n(x)\, dx = 0, \quad k = 0, \dots, n-2. \end{aligned} \tag{19}$$

Lemma 3.1. *The following relation*

$$r_n = -2^{-1} \int_0^1 \frac{\psi_n(x)}{1-x} dx. \tag{20}$$

holds.

Proof. In fact, we know that

$$\begin{aligned} -2^{-1} \int_0^1 \frac{\psi_n(x)}{1-x} dx &= -2^{-1} \int_0^1 \frac{A_n(x) - A_n(1)}{1-x} \log x dx \\ &\quad +2^{-1} \int_0^1 \frac{B_n(x) - B_n(1)}{1-x} \log^2 x dx \\ &\quad +2^{-1} B_n(1) \int_0^1 \frac{\log^2 x}{1-x} dx. \end{aligned}$$

Therefore, using (1) and (17) we get

$$\begin{aligned} -2^{-1}\int_0^1 \frac{\psi_n(x)}{1-x}dx &= q_n\zeta(3) \\ &\quad +2^{-1}\sum_{0\le k\le n} a_k^{(n)} \sum_{1\le j\le k}\int_0^1 x^{j-1}\log x dx \\ &\quad -2^{-1}\sum_{0\le k\le n} b_k^{(n)} \sum_{1\le j\le k}\int_0^1 x^{j-1}\log^2 x dx \\ &= q_n\zeta(3) - \sum_{0\le k\le n} b_k^{(n)} \sum_{1\le j\le k}\frac{1}{j^3} - 2^{-1}\sum_{0\le k\le n} a_k^{(n)}\sum_{1\le j\le k}\frac{1}{j^2}. \end{aligned}$$

Thus, the lemma is completely proved. □

Observe that, as a consequence of the conditions of orthogonality (19) and the Lemma 3.1 we deduce the following relations

$$\begin{aligned} \int_0^1 \frac{p_n(x)\varphi_n(x)}{1-x}dx &= p_n(1)\int_0^1 \frac{\varphi_n(x)}{1-x}dx, \\ \tilde{p}_{n-1}(1)\, r_n &= -2^{-1}\int_0^1 \frac{\tilde{p}_{n-1}(x)\psi_n(x)}{1-x}dx, \end{aligned} \qquad (21)$$

where $p_n(x)$ and $\tilde{p}_{n-1}(x)$ are arbitrary polynomials of degree at most n and $n-1$ respectively.

Proposition 3.2. *The following relations are valid*

i.)

$$p_n q_{n+1} - p_{n+1} q_n = -\frac{24n^3+30n^2+16n+3}{2n^4(n+1)^4}, \qquad (22)$$

ii.)

$$\zeta(3) = \frac{7}{6} + \sum_{n\ge 1}\frac{24n^3+30n^2+16n+3}{2n^3(n+1)^3\,\Theta_n}, \qquad (23)$$

being

$$\Theta_n = {}_4F_3\left(\left.\begin{array}{c} -n-1,-n-1,n+1,n+2 \\ 1,1,1 \end{array}\right| 1\right) \times {}_4F_3\left(\left.\begin{array}{c} -n,-n,n,n+1 \\ 1,1,1 \end{array}\right| 1\right),$$

where ${}_rF_s$ denotes the ordinary hypergeometric series [13, 17, 21].

Proof. Notice that, using (21) we have

$$\begin{aligned} q_n r_{n+1} &= 2^{-1} A_n(1) \int_0^1 \frac{\varphi_{n+1}(x)}{1-x} dx - 2^{-1} \int_0^1 \frac{B_n(x)\psi_{n+1}(x)}{1-x} dx \\ &= 2^{-1} \int_0^1 \frac{\varphi_n(x)\varphi_{n+1}(x)}{1-x} dx. \end{aligned}$$

Moreover

$$2^{-1} \int_0^1 \frac{\varphi_n(x)\varphi_{n+1}(x)}{1-x} dx = 2^{-1} \int_0^1 \frac{A_{n+1}(x)\varphi_n(x)}{1-x} dx - 2^{-1} \int_0^1 \frac{B_{n+1}(x)\psi_n(x)}{1-x} dx,$$

being

$$\begin{aligned} \int_0^1 \frac{A_{n+1}(x)\varphi_n(x)}{1-x} dx &= \int_0^1 \frac{[A_{n+1}(x) - A_{n+1}(1)]\varphi_n(x)}{1-x} dx \\ &= -\sum_{0\le k\le n+1} a_k^{(n+1)} \sum_{1\le j\le k} \int_0^1 x^{j-1}\varphi_n(x)\,dx \\ &= -a_{n+1}^{(n+1)} \mathcal{F}_n(n), \end{aligned}$$

and

$$\begin{aligned} -2^{-1} \int_0^1 \frac{B_{n+1}(x)\psi_n(x)}{1-x} dx &= -2^{-1} B_{n+1}(1) \int_0^1 \frac{\psi_n(x)}{1-x} dx \\ &\quad - 2^{-1} \int_0^1 \frac{[B_{n+1}(x) - B_{n+1}(1)]\psi_n(x)}{1-x} dx \\ &= 2^{-1} \sum_{0\le k\le n+1} b_k^{(n+1)} \sum_{1\le j\le k} \int_0^1 x^{j-1}\psi_n(x)\,dx + q_{n+1} r_n. \end{aligned}$$

From where we get

$$\begin{aligned} p_n q_{n+1} - p_{n+1} q_n &= q_n r_{n+1} - q_{n+1} r_n \\ &= 2^{-1} b_n^{(n+1)} \mathcal{G}_n(n-1) + 2^{-1} b_{n+1}^{(n+1)} \mathcal{G}_n(n-1) \\ &\quad + 2^{-1} b_{n+1}^{(n+1)} \mathcal{G}_n(n) - 2^{-1} a_{n+1}^{(n+1)} \mathcal{F}_n(n), \end{aligned}$$

which corresponds with (22). Presently, having in account

$$\frac{p_n}{q_n} = \frac{p_1}{q_1} - \sum_{1\le k\le n-1} \left(\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} \right),$$

and using (22) conjointly with

$$\zeta(3) = \lim_{n\to\infty} \frac{p_n}{q_n} = \frac{p_1}{q_1} - \sum_{n\ge 1} \left(\frac{p_n q_{n+1} - p_{n+1} q_n}{q_n q_{n+1}} \right).$$

We deduce (23). Therefore, the proposition is completely proved. □

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References

[1] Abramov, S. A. Applicability of Zeilberger's algorithm to hypergeometric terms, *Proc. of the International Symposium on Symbolic and Algebraic Computation (ISSAC'02)*, New York, 2002, 1–7.

[2] Abramov, S. A., H. Q. Le, A criterion for the applicability of Zeilberger's algorithm to rational functions, *Discrete Math.*, Vol. 259, 2002, 1–17.

[3] Abramov, S. A. When does Zeilberger's algorithm succeed?, *Appl. Math.*, Vol. 30, 2003, 424–441.

[4] Abramov, S. A., J. J. Carette, K. O. Geddes, H. Q. Le. Telescoping in the context of symbolic summation in Maple, *J. of Symb. Comput.*, Vol. 30, 2004, 1303–1326.

[5] Apéry, R. Irrationalité de $\zeta(2)$ et $\zeta(3)$, *Astérisque*, Vol. 61, 1979, 11–13.

[6] Arvesú, J. Orthogonal forms: A key tool for deducing Apéry's recurrence relation, *J. Approx. Theory*, accepted, 2012.

[7] Beukers, F. A note on the irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.*, Vol. 11, 1979, 268–272.

[8] Beukers, F. Legendre polynomials in irrationality proofs, *Bull. Austral. Math. Soc.*, Vol. 22, 1980, 431–438.

[9] Beuker, F. Padé approximations in number theory, *Padé approximation and its applications, (Amsterdam, 1980), 90–99, Lecture Notes in Math., Springer, Berlin-New York*, Vol. 888, 1981.

[10] Beuker, F. Consequences of Apéry's work on $\zeta(3)$, *preprint of talk presented at the Rencontres Arithmétiques de Caen, $\zeta(3)$, Irrationnel: Les Retombées*, 1995.

[11] Cohen, H. Demonstration de l'irrationalite de $\zeta(3)$ (d'aprés R. Apéry), *Séminaire de Théorie des Nombres*, Grenoble, 1978, VI.1–VI.9.

[12] Elsner, C. On Prime-Detecting Sequences From Apéry's Recurrence Formulae for $\zeta(3)$ and $\zeta(2)$, *J. Integer Sequences*, Vol. 11, 2008.

[13] Gasper, G., M. Rahman. Basic Hypergeometric Series, *Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge*, 2004.

[14] Gutnik, L. A. On the irrationality of certain quantities involving $\zeta(3)$, *Acta Arith.*, Vol. 42, 1983, 255–264.

[15] Jones, W. B., W. J. Thron. *Continued fractions*, Analytic theory and applications, Encyclopedia Math. Appl. Section: Analysis 11, Addison-Wesley, London, 1980.

[16] Kaneko, M., N. Kurokawa, M. Wakayama. A variation of Euler's approach to values of the Riemann zeta function, *arXiv:math/0206171v2 [math.QA]*, 31 July 2012.

[17] Koekoek, R., R. F. Swarttouw. The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, *Report 98-17, Faculty of Technical Mathematics and Informatics, Delft University of Technology*, 1998.

[18] Markoff, A. A. Mémoire sur la transformation des séries peu convergentes en séries tres convergentes, *Mémoires de l'Academie Impériale des Sciences de St.-Petersbourg, VII série, t. XXXVII*, No. 9, 1890.

[19] Nesterenko, Y. V. A few remarks on $\zeta(3)$, *Math. Notes*, Vol. 59, 1996, No. 6, 625–636.

[20] Nesterenko, Y. V. Integral identities and constructions of approximations to zeta values, *J. Théor. Nombres Bordeaux*, Vol. 15, 2003, 535–550.

[21] Nikiforov, A. F., V. B. Uvarov. Special Functions in Mathematical Physics, *Birkhauser Verlag, Basel*, 1988.

[22] Perron, O. Über ein Satz des Herrn Poincaré, *J. Reine Angew. Math., Über die Poincarésche lineare Differenzgleichung*, Vol. 137, 1910, 6–64.

[23] Poincaré, H., Sur les équations linéaires aux différentielles et aux différences finies, *Amer. J. Math.*, Vol. 7, 1885, 203–258.

[24] Prévost, M. A new proof of the irrationality of $\zeta(2)$ and $\zeta(3)$ using Padé approximants, *J. Comput. Appl. Math.*, Vol. 67, 1996, 219–235.

[25] Petkovsek, M., H. S. Wilf, D. Zeilberger, $A = B$, *A. K. Peters, Ltd., Wellesley, M.A.*, 1997.

[26] Sorokin, V. N. Hermite-Padé approximations for Nikishin systems and the irrationality of $\zeta(3)$, *Communications of the Moscow Math. Soc.*, Vol. 49, 1993, 176–177.

[27] Sorokin, V. N. Apéry's theorem, *Vestnik Moskov. Univ. Ser. I Mat. Mekh. [Moscow Univ. Math. Bull.]*, No. 3, 1998, 48–52.

[28] Van der Poorten, A. A proof that Euler missed... Apéry's proof of the irrationality of $\zeta\,(3)$, *Math. Intelligencer*, Vol. 1, 1978/79, 195–203.

[29] Van Assche, W. Multiple orthogonal polynomials, irrationality and transcendence, *Contemporary Mathematics*, Vol. 236, 1999, 325–342.