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On a recurrence related to $321-$avoiding permutations

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Abstract: Dokos et al. recently conjectured that the distribution polynomial $f_n(q)$ on the set of permutations of size n avoiding the pattern 321 for the number of inversions is given by

$$f_n(q) = f_{n-1}(q) + \sum_{k=0}^{n-2} q^{k+1} f_k(q) f_{n-1-k}(q), \qquad n \geq 1,$$

with $f_0(q) = 1$, which was later proven in the affirmative, see [1]. In this note, we provide a new proof of this conjecture, based on the scanning-elements algorithm described in [3], and present an identity obtained by equating two explicit formulas for the generating function $\sum_{n\geq 0} a_n(q)x^n$.
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1 Introduction

Let S_n denote the set of permutations of size n. Two sequences of distinct numbers, $a = a_1a_2\cdots a_n$ and $b = b_1b_2\cdots b_n$, are said to be *order isomorphic* whenever they satisfy $a_i > a_j$ if and only if $b_i > b_j$, for all $1 \leq i < j \leq n$. We will say that $\pi \in S_n$ contains $\tau \in S_k$ as a *pattern* if there is a subsequence of π that is order isomorphic to τ. For example, the permutation $\pi = 14523 \in S_5$ contains the pattern 132 but not the pattern 213. We denote the subset of permutations of S_n whose members do not contain (i.e., *avoid*) the pattern τ by $S_n(\tau)$. The *inversion* *number* of $\pi = \pi_1\pi_2\cdots\pi_n$, denoted $inv(\pi)$, is the number of occurrences of the pattern 21 in π, that is,

$$inv(\pi) = |\{(i,j) \mid \pi_i > \pi_j \text{ and } 1 \leq i < j \leq n\}|.$$

Dokos et al. [2] conjectured that the polynomial $f_n(q) = \sum_{\pi \in S_n(321)} q^{inv(\pi)}$ satisfies

$$f_n(q) = f_{n-1}(q) + \sum_{k=0}^{n-2} q^{k+1} f_k(q) f_{n-1-k}(q), \qquad n \geq 1, \tag{1}$$

with $f_0(q) = 1$. More recently, this conjecture has been shown to be true by Cheng et al. [1, Theorem 2.2], who used a bijection together with previous combinatorial results from [4] concerning the distribution of some statistics on parallelogram polyominoes and ballot sequences. In this note, we hope to shed some further light on this recurrence and provide a different solution for it, one that is more algebraic. Our main result may be formulated as follows, where we have used the notation $(x;q)_j = (1-x)(1-qx)\cdots(1-q^{j-1}x)$.

Theorem 1. *The generating function $\sum_{n\geq 0} f_n(q)x^n$ is given by*

$$1 + \frac{\sum_{j\geq 0} \frac{(-1)^j q^{\binom{j+1}{2}} x^{j+1}}{(q;q)_j (xq;q)_{j+1}}}{\sum_{j\geq 0} (-1)^j \frac{q^{\binom{j}{2}} x^j}{(q;q)_j (xq;q)_j}} = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{xq}{1 - \cfrac{xq}{1 - \cfrac{xq^2}{1 - \cfrac{xq^2}{\ddots}}}}}}.$$

Moreover, the polynomial $f_n(q)$ satisfies (1).

The proof of this theorem is given in the next section and is based on the scanning-elements algorithm described in [3].

2 Proof of Theorem 1

Let $f_n(q|a)$ denote the generating function which counts the members $\pi = \pi_1\pi_2\cdots\pi_n \in S_n(321)$, with $\pi_1 = a$, according to the number of inversions, that is,

$$f_n(q|a) = \sum_{\pi = a\pi' \in S_n(321)} q^{inv(\pi)}.$$

Clearly, $f_n(q) = \sum_{a=1}^{n} f_n(q|a)$. From the definitions, we can state the recurrence

$$f_n(q|a) = qf_{n-1}(q|a-1) + q^{a-1} \sum_{j=a}^{n-1} f_{n-1}(q|j), \quad 2 \leq a \leq n,$$

with $f_n(q|1) = f_{n-1}(q)$. To solve this recurrence, we let $F_n(q;t) = \sum_{a=1}^{n} f_n(q|a)t^{a-1}$. Multiplying the above recurrence relation by t^{a-1}, and summing over $a = 2, 3, \ldots, n$, we obtain

$$F_n(q;t) = \sum_{j=1}^{n-1} \frac{qt-(qt)^j}{1-qt} f_{n-1}(q|j) + qtF_{n-1}(q;t) + F_{n-1}(q;1),$$

which is equivalent to

$$F_n(q;t) = \frac{qt}{1-qt}(F_{n-1}(q;1) - F_{n-1}(q;qt)) + qtF_{n-1}(q;t) + F_{n-1}(q;1), \qquad n \geq 2,$$

with $F_1(q;t) = 1$.

Now define $F(x,q;t) = \sum_{n\geq 1} F_n(q;t)x^n$. Then the last recurrence may be expressed as

$$F(x,q;t) = \frac{x}{1-qtx} + \frac{x}{(1-qt)(1-qtx)}F(x,q;1) - \frac{qtx}{(1-qt)(1-qtx)}F(x,q;qt).$$

Iterating this equation an infinite number of times, we obtain

$$F(x,q;t) =$$

$$= \sum_{j\geq 0}(-1)^j \left(\frac{x}{1-q^{j+1}tx} + \frac{x}{(1-q^{j+1}t)(1-q^{j+1}tx)}F(x,q;1)\right) \prod_{i=1}^{j} \frac{q^i tx}{(1-q^i t)(1-q^i tx)},$$

which gives

$$F(x,q;t) = \sum_{j\geq 0}(-1)^j \left(1-q^{j+1}t + F(x,q;1)\right) \frac{q^{\binom{j+1}{2}} t^j x^{j+1}}{\prod_{i=1}^{j+1}(1-q^i t)(1-q^i tx)}.$$

Hence, we can state the following result.

Theorem 2. *We have*

$$F(x,q;1) = \frac{\sum_{j\geq 0} \frac{(-1)^j q^{\binom{j+1}{2}} x^{j+1}}{(q;q)_j (xq;q)_{j+1}}}{\sum_{j\geq 0}(-1)^j \frac{q^{\binom{j}{2}} x^j}{(q;q)_j (xq;q)_j}}.$$

To prove (1), we will show that the sequence determined by (1) has generating function $1 + F(x,q;1)$. To do so, we define

$$\frac{A(x,q)}{B(x,q)} \equiv \frac{\sum_{j\geq 0} \frac{(-1)^j q^{\binom{j+1}{2}} x^{j+1}}{(q;q)_j (xq;q)_{j+1}}}{\sum_{j\geq 0}(-1)^j \frac{q^{\binom{j}{2}} x^j}{(q;q)_j (xq;q)_j}}.$$

Theorem 2 states that $F(x, q; 1) = \dfrac{A(x, q)}{B(x, q)}$. The next lemma supplies the key properties of $A(x, q)$ and $B(x, q)$ which we'll need.

Lemma 3. *We have*

$$
\begin{aligned}
B(xq, q) &= \frac{1 - xq}{x} A(x, q), \\
A(xq, q) &= \frac{1 - xq}{x^2 q}\big((1 - x - xq)A(x, q) - xB(x, q)\big).
\end{aligned}
$$

Proof. From the definitions, we have

$$
B(xq, q) = \sum_{j \geq 0} (-1)^j \frac{q^{\binom{j}{2}+j} x^j}{(q; q)_j (xq^2; q)_j} = \frac{1 - xq}{x} A(x, q)
$$

and

$$
\begin{aligned}
&(1 - x - xq)A(x, q) - xB(x, q) \\
&= (1 - x - xq) \sum_{j \geq 0} \frac{(-1)^j q^{\binom{j+1}{2}} x^{j+1}}{(q; q)_j (xq; q)_{j+1}} - x \sum_{j \geq 0} (-1)^j \frac{q^{\binom{j}{2}} x^j}{(q; q)_j (xq; q)_j} \\
&= \sum_{j \geq 0} \frac{(-1)^j x^{j+1} q^{\binom{j}{2}}}{(q; q)_j (xq; q)_{j+1}} \big((1 - x - xq)q^j - 1 + xq^{j+1}\big) \\
&= \sum_{j \geq 0} \frac{(-1)^j x^{j+1} q^{\binom{j}{2}}}{(q; q)_j (xq; q)_{j+1}} \big((1 - x)q^j - 1\big) \\
&= \sum_{j \geq 0} \frac{(-1)^j x^{j+2} q^{\binom{j+1}{2}}}{(q; q)_j (xq; q)_{j+2}} - \sum_{j \geq 0} \frac{(-1)^j x^{j+2} q^{\binom{j+1}{2}}}{(q; q)_j (xq; q)_{j+1}} \\
&= \sum_{j \geq 0} \frac{(-1)^j x^{j+3} q^{\binom{j+2}{2}+1}}{(q; q)_j (xq; q)_{j+2}} \\
&= \frac{x^2 q}{1 - xq} A(xq, q),
\end{aligned}
$$

as required. □

Lemma 4. *Let a_n be the sequence defined by the recurrence*

$$
a_n = a_{n-1} + \sum_{k=0}^{n-2} q^{k+1} a_k a_{n-1-k}, \qquad n \geq 1,
$$

with $a_0 = 1$. Then the generating function $\sum_{n \geq 0} a_n x^n$ is given by $1 + F(x, q; 1)$.

Proof. Let $G(x) = \sum_{n \geq 0} a_n x^n$. Then the recurrence may be written as

$$
G(x) = 1 + xG(x) + xqG(x)G(xq) - xqG(xq),
$$

which implies

$$G(x) = \frac{1 - xqG(xq)}{1 - x - xqG(xq)}. \tag{2}$$

On the other hand,

$$\begin{aligned} -1 + \frac{1 - xq(F(xq, q; 1) + 1)}{1 - x - xq(F(xq, q; 1) + 1)} &= \frac{x}{1 - x - xq(F(xq, q; 1) + 1)} \\ &= \frac{x}{1 - x - xq - xq\dfrac{A(xq, q)}{B(xq, q)}}, \end{aligned}$$

so by Lemma 3, we have

$$-1 + \frac{1 - xq(F(xq, q; 1) + 1)}{1 - x - xq(F(xq, q; 1) + 1)} = \frac{A(x, q)}{B(x, q)} = F(x, q; 1).$$

Note that the functional equation (2) has a unique power series solution since it determines the coefficients of such a solution. Hence by uniqueness, $G(x) = F(x, q; 1) + 1$. □

Theorem 1 now follows from Theorem 2 and Lemma 4.

References

[1] Cheng, S.-E., S. Elizalde, A. Kasraoui, B. E. Sagan. Inversion and major index polynomials, Preprint, `http://arxiv.org/pdf/1112.6014.pdf`.

[2] Dokos, T., T. Dwyer, B. P. Johnson, B. E. Sagan, K. Selsor. Permutation patterns and statistics, *Discrete Math.*, Vol. 312, 2012, 2760–2775.

[3] Firro, G., T. Mansour. Three-letter-pattern-avoiding permutations and functional equations, *Electron. J. Combin.*, Vol. 13, 2006, #R51.

[4] Fürlinger, J., J. Hofbauer, q-Catalan numbers, *J. Combin. Theory Ser. A*, Vol. 40, 1985, No. 2, 248–264.